Ramanujan the man and mathematics

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It is about the life and works of India's greatest Mathematician Srinivasa Ramanujan. Formally, less trained as he was, did most of his mathematics on his own until he found Hardy a celebrated Cambridge based Mathematician. G.H.Hardy encouraged Ramanujan by inviting him to Cambridge when the latter wrote to him of his works and pleading for help. Ramanujan Srinivasa Iyengar in short S.Ramanujan as he was commonly known to most of us was born to a traditional South Indian couple at Erode in Tamilnadu on December 22 1887.His father Srinivasa was working for a cloth merchant in Kumbakonam a temple town located 170miles away from Madras. Ramanujan grew up and went to School at this place. His school record was impressive. A creditable pass at School final examination also earned him a scholarship for his college studies. Not much is known of his early life except for his excessive interest in Mathematics neglecting there by the other subjects which led him stand in eligible to the formal University Education and his marriage with Janaki Ammal were regarded as the only significant events. His wife it seems was just 9 years at the time of his marriage and he was almost twice as her age. In those days the girls were discouraged from studying but were trained in the house hold skills and so that it would help them to manage the home front chores. His mother was a great influence on him and he was guided by her right from his early life till he left for England. Generally, a Hindu family used to be a undivided one and elders would decide what is important for their children on all most all the matters right from their formative years to the issues relating to marriage and beyond.

His house with a thatched roof with a veranda in the front was right on the Sarangapani Sannidhi Street. Sarangapani is a Hindu God (Lord Vishnu, Sannidhi is a Sanskrit term meaning abode and Street of course is the English term, that is the legacy that British left on India which came in good stud).Ramanujan was deeply religious but Hardy notes that it never came in the way of his professional life .It seems that Hardy used to say that Ramanujan was his discovery and one of the most romantic experiences of his professional life. Hardy made genuine efforts in bringing Ramanujan to Cambridge and gave him formal expositions to all that Mathematics he missed while in India. In the year 1912 he moved to Madras. Thanks to Madras Port Trust Office which offered him a clerical post and also that proved to be a turning point for his professional career.

Ramanujan was encouraged by his friends to communicate his work and he was also interested in ascertaining from the people for an opinion on his work. It seems that he had corresponded to Cambridge Mathematicians before he wrote to Hardy. Unfortunately, both of them turn down Ramanujan's work. But this was not the case with Hardy, although initially he was not very clear about what all Ramanujan was trying to claim but he took the help of his collaborator friend J.E. Littlewood and both after taking a close look at the contents of the letter he communicated came to a conclusion that this is a work of a genius and deserves every encouragement. Later, Hardy without losing any more time wrote back and thus inviting Ramanujan to Cambridge. Not only that he worked hard which almost took one year to see that Ramanujan accepted his invitation to join him later in the year 1914 when Ramanujan ,finally set his journey to England. When Ramanujan got a letter of invitation from Hardy to visit Cambridge, Ramanujan was in a fix at the first instance his mother who was such an influence on Ramanujan declined permission and more over around same time the University of Madras awarded Ramanujan a Fellow Ship(was to commence from May 1914) under these circumstances he wrote back Hardy for disappointing him by not accepting his invitation and leave for England. As we all know he left for England during January 1914 and the rest is History. Hardy gave some lectures to Ramanujan which covered a good deal of Mathematics all that he missed. The climate, food and social life at Cambridge was unusual to him and he had lots of problems and get adjusted to those life styles. He fell sick quite frequently and after some stage lived with it. As his condition got deteriorated it was thought safe of sending him back to India, which also got delayed on account of the war. However , he sailed back to India in the year 1919 and one year later he succumbed to his illness which was some time diagnosed as tuberculosis but it was not so as it was thought to be.

On his Notebooks:

It was clear that Ramanujan was writing his mathematical works in the form of notebooks during the years 1903-1914. They were in all three; the first notebook was left with Hardy when he returned to India. The remaining two notebooks (second and third) were donated to the library of the University of Madras and the one that was with Hardy was also returned to the library through S.R.Ranganathan who was the librarian of Madras University. (S.R.Ranganathan was visiting Cambridge at that time and Hardy handed it over to him)

The University of Madras photocopied all the three handwritten notebooks of Ramanujan and a copy of each was sent to Hardy [3].Hardy and others strongly urged the notebooks to be edited and published. In the year 1929 Prof G.N.Watson and Prof B.M.Wilson undertook the task of editing the notebooks. Wilson unfortunately passed away prematurely while Watson continued with the task till late 30's when he lost interest in the work. In the year 1957 the Tata Institute of Fundamental Research (TIFR) in Bombay published a Photostat edition of the notebooks in two volumes, [4] the first volume reproduces Ramanujan's first notebook while the second contains second and third notebooks. Again 80's saw the resurgence of Ramanujan's work coming to light largely due to the efforts of George. E. Andrews of Penn State University and Bruce Berndt of University of Illinois for editing Ramanujan's notebooks in three volumes published by Springer- Verlag,[3]. Ramanujan worked extensively on infinite series more so on divergent series note his contemporaries. Even though he was regarded as a Number theorist but his contribution to Analysis was no less.On his mathematics I, have considered his work on partition functions and its related arithmetic. Recently, K. Uno and others have came up with startling results on Ramanujan congruences,[1],[4] .

On Ramanujan Congruence

The following congruence is due to Ramanujan

$$P\ (5n+4) \equiv 0 \pmod 5$$

$$P\ (7n+5) \equiv 0 \pmod 7$$

$$P\ (11n+6) \equiv 0 \pmod{11}, \text{ where } P(n) \text{ denotes the partition function.}$$

Ramanujan took note of Mac Mahon's work in computing the values of p(n) for n=200 and carefully went through the tabulated values of them before he came up with his theorem on the partition congruence of n.

A partition of n is a non-increasing list of positive integers $\lambda_1 \geq \lambda_2 \geq \ldots\ldots\ldots\ldots \geq \lambda_k$ that sum to n; we write $|\lambda| = n$ .The partition function p(n) is defined to count the number of distinct partitions of a given integer n.

Watson and Atkin extended these congruences to arbitrary powers of 5,7 and 11, [2]. Some more progress in proving congruences for primes up to 31 was also witnessed, until a remarkable paper in this regard appeared. It was due to K.Ono who proved the existence of infinite families of partition congruences for every prime $l \geqq 5$, [3]; he used the idea of p-adic theory and modular forms. These results were much more complicated than Ramanujan's original congruences. But, they derive inspiration from the works of Freeman Dyson on Ramanujan congruences.

Freeman Dyson noticed some strange thing happening with Ramanujan congruences, with his 'Crank ' functions he formalized them and made a conjecture for the prime power moduli , see[6] .

If $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_k$ the partition λ of n, $|\lambda| = n$ then the rank of λ is defined as

$$\text{Rank}(\lambda) = \lambda_1 - k$$

We can compute ranks of p (4), p (5) and p (6). In case of these partitions he noticed that the partitions can be grouped into residue classes modulo those primes, [1].

If (m, N, n) is the number of partitions λ of n for which Rank (λ) ≡ m (modN) then he noticed that

$$(m, 5, 5n+4) = \frac{1}{5}\, p\,(5n+4)\ , \qquad 0 \leq m \leq 4$$

Later Atkin and Swinnerton –Dyer proved Dyson's observation,[6].However the rank of partition even for small prime modulus does not equally dissect the Ramanujan congruence modulo 11. As a result Dyson came across a strange function which he called a 'crank' as mentioned earlier for the congruences to exhibit the desired distribution. Dyson's assertion was cleared by George Andrews and Garvan almost forty years later. They defined even this crank function and showed that

$$\mathcal{M}\,(m, 11, 11n+6) = \frac{1}{11}\, p\,(11n+6)$$

$\mathcal{M}$ (m, $\mathcal{N}$, n) is defined for the crank just as (m, N, n) was for the rank, [5].

In these historic investigations, they also showed that the crank dissects the Ramanujan congruences modulo 5 and 7 in a different way than the rank, [7]. For the partition λ of n Crank (λ) is defined by

$$\text{Crank}(\lambda) = \begin{cases} \lambda_1 & \text{if } r = 0 \\ O(\lambda) - r & \text{if } r \geq 1 \end{cases}$$

Where λ = If $\lambda_1 + \lambda_2 + \ldots\ldots\ldots\ldots + \lambda_s + 1 + \ldots\ldots + 1$

With exactly r ones and O (λ) denotes the number of parts of λ that are strictly larger than r.

Clearly p (n) = $\mathcal{M}$ (0, N, n) + ... + $\mathcal{M}$ (N-1, N, n) and for the Ramanujan Congruences all of these summands are equal. A theorem due to Ono shows that the congruences for the partition function are related to the Crank in a universal fashion. In his unpublished result he conjectured that for every prime l ≥ 5, and $\tau \geq 1$ there are infinitely many non nested arithmetic progressions An+B for which $\mathcal{M}$ (m, l, An+B) = 0 mod l^t for every $0 \leq m \leq l-1$.

Further, the following theorem shows that the Crank function actually satisfies congruences beyond those predicted by Ono.

Theorem: Suppose that l ≥ 5 is prime and that τ and j ate positive integers. Then there are infinitely many non nested arithmetic progressions An+B such that

$$\mathcal{M}(m, l^j, An+B) = 0 \bmod l^{\tau} \text{ for every } 0 \leq m \leq l^j - 1.$$

Of late, developments in understanding the arithmetic of p(n) and results presented by Ono and others indicate that Ramanujan conjectures on partition functions are intimately related to modular theory and they assume a unified form for the partition congruences . By noting l as earlier we may see that the Ramanujan congruences for its prime power moduli can be stated as

$$p\ (An+B) = 0 \bmod l^{k}$$

For l =11 it has been noted that the progressions 11n+1, 11n+2, 11n+3, 11n+5, 11n+6 , 11n+8 exist s within each of (l+1)/2 modulo,[8].

It appears that Ramanujan works have found applications in polymer chemistry, computer science, and physics and even in cancer research. Ramanujan graphs (named after him because of its connection with Ramanujan conjecture) constructed by means of quaternion algebras have found applications in communication engineering characterizing certain efficient networks (graphs as expanders). We are working on Ramanujan graphs for some of our problems in coding theory, it involves constructing Ramanujan graphs which are expanders(remarkably connected),[9].

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