# A SMALL CORRECTION TO A PAPER OF VANDERMONDE

Matthew Davis and Shiv K. Gupta
Department of Mathematics
West Chester University

In an often quoted impressive paper Vandermonde [5] comes very close to introducing the concept of Cyclotomy and Galois Theory - some forty years before the birth of Galois. Discussing the work of Vandermonde Jean-Pierre Tignol [4] writes

> "$\cdots$The existence of a cyclic permutation which does preserve the relations (among the roots) is very remarkable; and quite mysterious property of cyclotomic equations, which should have awaken Vandermonde's curiosity. If he had investigated this property he could have developed the theory of cyclotomy about 30 years before Gauss."

The fact that Vandermonde did his investigations without the benefit of the knowledge of the work of Gauss (as Galois did) is indeed very noteworthy. His observation about the existence of cyclic permutation which preserve the relations among the roots is forerunner to the introduction of the concept of automorphism in later times.

In the last article of the aforementioned paper Vandermonde talks about applying the theory developed earlier to solve (that is to find radical expressions for the roots of) the equation $z^{11}-1=0$, having disposed of the similar problems for $z^n-1=0$, for smaller values of $n$ in the earlier articles of the same paper. In particular he gives radical expressions for $2\cos\frac{2k\pi}{11}$ for $1\le k\le 5$ (actually for some unknown reasons Vandermonde prefers to work with the equation whose roots are $-2\cos\frac{2k\pi}{11}$ for $1\le k\le 5$). Vandermonde does not provide the details of his calculations - he only points out to the method of doing so and gives the final result. Tignol [2] has reproduced the radical expressions for $-2\cos\frac{2k\pi}{11}$ for $1\le k\le 5$ as they appear in Vandermonde's paper - still not providing any of the details which led him (or Vandermonde) to the radical expressions for $-2\cos\frac{2k\pi}{11}$ for $1\le k\le 5$. These radical expression for $-2\cos\frac{2k\pi}{11}$ for $1\le k\le 5$, given by Vandermonde and reproduced from there by Tignol are:

Let

$$a=-2\cos\frac{2\pi}{11},\quad b=-2\cos\frac{4\pi}{11},\quad c=-2\cos\frac{6\pi}{11},\quad d=-2\cos\frac{8\pi}{11},\quad e=-2\cos\frac{10\pi}{11}$$

then

$$a, b, c, d, e = \frac{1}{5}\left[1 + \Delta^{'} + \Delta^{''} + \Delta^{'''} + \Delta^{iv}\right]$$

where

$$\Delta^{'} = \sqrt[5]{\frac{11}{4}\left(89 + 25\sqrt{5} - 5\sqrt{-5 + 2\sqrt{5}} + 45\sqrt{-5 - 2\sqrt{5}}\right)}$$

$$\Delta^{''} = \sqrt[5]{\frac{11}{4}\left(89 + 25\sqrt{5} + 5\sqrt{-5 + 2\sqrt{5}} - 45\sqrt{-5 - 2\sqrt{5}}\right)}$$

$$\Delta^{'''} = \sqrt[5]{\frac{11}{4}\left(89 - 25\sqrt{5} - 5\sqrt{-5 + 2\sqrt{5}} - 45\sqrt{-5 - 2\sqrt{5}}\right)}$$

$$\Delta^{iv} = \sqrt[5]{\frac{11}{4}\left(89 - 25\sqrt{5} + 5\sqrt{-5 + 2\sqrt{5}} + 45\sqrt{-5 - 2\sqrt{5}}\right)}$$

Despite our repeated efforts we failed to verify the correctness of above expressions. We decided to evaluate the expressions for $\Delta^{'}, \Delta^{''}, \Delta^{'''}$, and $\Delta^{iv}$ first by hand calculations (using the reductions procedures indicated in Vandermonde's paper) and later by the help of MATHEMATICA. We claim that the correct expressions for $\Delta^{'}, \Delta^{''}, \Delta^{'''}$, and $\Delta^{iv}$ are:

$$\Delta_c^{'} = \sqrt[5]{\frac{11}{4}\left(89 + 25\sqrt{5} + \mathbf{20\sqrt{-10 - 2\sqrt{5}} - 25\sqrt{-10 + 2\sqrt{5}}}\right)}$$

$$\Delta_c^{''} = \sqrt[5]{\frac{11}{4}\left(89 + 25\sqrt{5} - \mathbf{20\sqrt{-10 - 2\sqrt{5}} + 25\sqrt{-10 + 2\sqrt{5}}}\right)}$$

$$\Delta_c^{'''} = \sqrt[5]{\frac{11}{4}\left(89 - 25\sqrt{5} - \mathbf{25\sqrt{-10 - 2\sqrt{5}} - 20\sqrt{-10 + 2\sqrt{5}}}\right)}$$

$$\Delta_c^{iv} = \sqrt[5]{\frac{11}{4}\left(89 - 25\sqrt{5} + \mathbf{20\sqrt{-10 - 2\sqrt{5}} + 25\sqrt{-10 - 2\sqrt{5}}}\right)}$$

In the above the corrections have been indicated by the boldfaced parts of the expressions.

The contribution of Vandermonde to the solutions of equations predates the work of Lagrange on the same by some twenty years. The work of Vandermonde and the opinion of Lagrange about it have also been mentioned in the article by Robin Rider Hamburg [3]. Specifically, while discussing cyclotomic equation Lagrange [2] makes the following statement about solving the equation

$$u^5 + u^4 - 4u^3 - 3u^2 + 3u + 1 = 0$$

> En prenant $u$ négativement, ce qui change les signes de tous les termes pairs, on a l'equation résolve par Vandermonde. Cet Auteur ne donne l'éxpression dont s'agit que comme un résultat de sa methode génerale, sans indiquer en détail les opérations par lesquelles il y est parvenu, et personne, aprés lui, ne s'est occupé, que je sache, à vérifer ce résultat, que paraît même éste ignoré.

which among other things essentially says that by replacing $u$ by $-u$ one obtains the equation solved by Vandermonde, namely $u^5 - u^4 - 4u^3 + 3u^2 + 3u - 1 = 0$, and he does not know of anyone who has verified the results obtained by Vandermonde. In this paper we are trying to do just that.

In the explanation that follow we shall substantially adhere to the notations used by Vandermonde and Tignol. However at some places we shall use the notions from elementary Galois theory for the justification for some of the steps in the calculations.

We begin by considering the equation

$$z^{11} - 1 = 0 \tag{1}$$

Dividing the above by $z - 1$ we get the equation

$$z^{10} + z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0 \tag{2}$$

whose roots are

$$e^{\frac{2\pi ik}{11}} = \cos\frac{2\pi k}{11} + i\sin\frac{2\pi}{11}, \qquad 1 \leq k \leq 10$$

Dividing this equation by $z^5$ and making the substitution $z + \dfrac{1}{z} = y$ we get

$$y^5 + y^4 - 4y^3 - 3y^2 + 3y + 1 = 0 \tag{3}$$

whose five roots are:

$$e^{\frac{2\pi ik}{11}} + e^{-\frac{2\pi ik}{11}} = 2\cos\frac{2\pi k}{11}, \qquad \text{for} \quad 1 \leq k \leq 5.$$

We would have preferred to work with equation (3). However as indicated earlier, for some reasons Vandermonde replaces $y$ by $-x$ in (3). In order to provide a convenient comparison with Vandermonde's calculations we shall do likewise. So by replacing $y$ by $-x$ in (3) we get the equation

$$x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 = 0 \tag{4}$$

whose five roots are: $-2\cos\frac{2\pi k}{11}$, for $1 \leq k \leq 5$.

If we let $\alpha = e^{\frac{2\pi i}{11}}$, the mapping $\alpha \to \alpha^2$, which permutes the 10 roots of equation (2) as

$$\alpha \to \alpha^2 \to \alpha^4 \to \alpha^8 \to \alpha^5 \to \alpha^{10} \to \alpha^9 \to \alpha^7 \to \alpha^3 \to \alpha^6 \to \alpha$$

gives the generating automorphism of the field $Q(\alpha)$. The restriction of this mapping to the subfield $Q(\cos\frac{2\pi}{11})$ (its maximum real subfield) induces an automorphism of $Q(\cos\frac{2\pi}{11})$. To see the effect of this automorphism on $Q(\cos\frac{2\pi}{11})$ we write:

$$\begin{aligned}
a &= -2\cos\frac{2\pi}{11} = -(\alpha + \frac{1}{\alpha}) = -(\alpha + \alpha^{10}) \\
b &= -2\cos\frac{4\pi}{11} = -(\alpha^2 + \frac{1}{\alpha^2}) = -(\alpha^2 + \alpha^9) \\
c &= -2\cos\frac{6\pi}{11} = -(\alpha^3 + \frac{1}{\alpha^3}) = -(\alpha^3 + \alpha^8) \\
d &= -2\cos\frac{8\pi}{11} = -(\alpha^4 + \frac{1}{\alpha^4}) = -(\alpha^4 + \alpha^7) \\
e &= -2\cos\frac{10\pi}{11} = -(\alpha^5 + \frac{1}{\alpha^5}) = -(\alpha^5 + \alpha^6)
\end{aligned}$$

Consider the tower of fields:

$$\begin{tikzcd}
Q(\alpha) \arrow[d, dash, "2"'] \\
Q(\cos\frac{2\pi}{11}) \arrow[d, dash, "5"'] \\
Q
\end{tikzcd}$$

The automorphism of the field $Q(\alpha)$ given by $\alpha \to \alpha^2$ induces an automorphism of its subfield $Q(\cos \frac{2\pi}{11})$ which cyclically permutes the five roots of (3) as follows:

$$\alpha + \alpha^{-1} \to \alpha^2 + \alpha^{-2} \to \alpha^4 + \alpha^{-4} \to \alpha^8 + \alpha^{-8} \to \alpha^5 + \alpha^{-5} \to \alpha^{10} + \alpha^{-10} = \alpha + \alpha^{-1}$$

or

$$2\cos\frac{2\pi}{11} \to 2\cos\frac{4\pi}{11} \to 2\cos\frac{8\pi}{11} \to 2\cos\frac{6\pi}{11} \to 2\cos\frac{10\pi}{11}, \to 2\cos\frac{2\pi}{11}$$

which is same as

$$a \to b \to d \to c \to e \to a$$

In other words the cyclic permutation $(a, b, d, c, e)$ preserves the (rational) relations among the roots of the equation (3). In the words of Tignol

> Vandermonde's brilliant (but not quite explicit) observation is that the permutation $a \to b \to d \to c \to e \to a$ preserves the relations between the roots.

As $(\alpha + \alpha^{-1})^2 = \alpha^2 + \alpha^{-2} + 2$, we have $a^2 = -b + 2$ (which is same as $\cos\frac{4\pi}{11} = 2\cos^2\frac{2\pi}{11} - 1$). Applying the permutation $(a, b, d, c, e)$ to the relation

$$a^2 = -b + 2$$

we get

$$b^2 = -d + 2$$

(which is same as $\cos\frac{8\pi}{11} = 2\cos^2\frac{4\pi}{11} - 1$).

Also as

$$(\alpha + \frac{1}{\alpha}) \cdot (\alpha^2 + \frac{1}{\alpha^2}) = (\alpha^3 + \frac{1}{\alpha^3}) + (\alpha + \frac{1}{\alpha})$$

which is same as saying

$$2\cos\frac{2\pi}{11} \cdot 2\cos\frac{4\pi}{11} = 2\cos\frac{6\pi}{11} + 2\cos\frac{2\pi}{11}$$

or

$$a \cdot b = -a - c$$

which is one of the 15 relations (among $a, b, c, d$, and $e$) that Vandermonde indicates for the process of reduction (to be explained soon). The following is an easily verifiable list of relations among $a, b, c, d$ and $e$ given by Vandermonde

$$\begin{array}{llll}
a^2 = -b + 2 & a \cdot b = -a - c & b \cdot c = -a - e & c \cdot d = -a - d \\
b^2 = -d + 2 & a \cdot c = -b - d & b \cdot d = -b - e & c \cdot e = -b - c \\
c^2 = -e + 2 & a \cdot d = -c - e & b \cdot e = -c - d & d \cdot e = -a - b \\
d^2 = -c + 2 & a \cdot e = -d - e & & \\
e^2 = -a + 2 & & &
\end{array}$$

Since the permutation $(a, b, d, c, e)$ induces the permutations

$$a \cdot b \to b \cdot d \to d \cdot c \to c \cdot e \to e \cdot a \to a \cdot b$$

$$a \cdot c \to b \cdot e \to d \cdot a \to c \cdot b \to e \cdot d \to a \cdot c$$

all the 15 relations among $a, b, c, d$ and $e$ given by Vandermonde can be derived by applying the permutation $(a, b, d, c, e)$ to the one of the three relations, namely

$$a^2 = -b + 2, \qquad a \cdot b = -a - c, \qquad a \cdot c = -b - d$$

It can be easily seen that any polynomial in $a, b, c, d$ and $e$ with rational coefficients can be reduced to a linear polynomial in the same by using the 15 relations among them given by Vandermonde. We refer to this procedure as the *process of reduction*.

Let $\omega = e^{\frac{2\pi i}{5}} = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$ be a primitive fifth root of unity. It is easy to see that

$$\cos \frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}, \qquad \sin \frac{2\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

For a later use we shall write down explicitly the values of $\omega^j$ for $1 \leq j \leq 4$.

$$\omega = e^{\frac{2\pi i}{5}} = \frac{-1 + \sqrt{5}}{4} + \frac{\sqrt{10 + 2\sqrt{5}}}{4} i$$

$$\omega^2 = e^{\frac{4\pi i}{5}} = \frac{-1 - \sqrt{5}}{4} + \frac{\sqrt{10 - 2\sqrt{5}}}{4} i$$

$$\omega^3 = e^{\frac{6\pi i}{5}} = \frac{-1-\sqrt{5}}{4} - \frac{\sqrt{10-2\sqrt{5}}}{4} i$$

$$\omega^4 = e^{\frac{8\pi i}{5}} = \frac{-1+\sqrt{5}}{4} - \frac{\sqrt{10+2\sqrt{5}}}{4} i$$

Since $a, b, c, d$ and $e$ are roots of (4) we also have

$$a + b + c + d + e = 1$$

Let $V_1, V_2, V_3$ and $V_4$ be defined as follows:

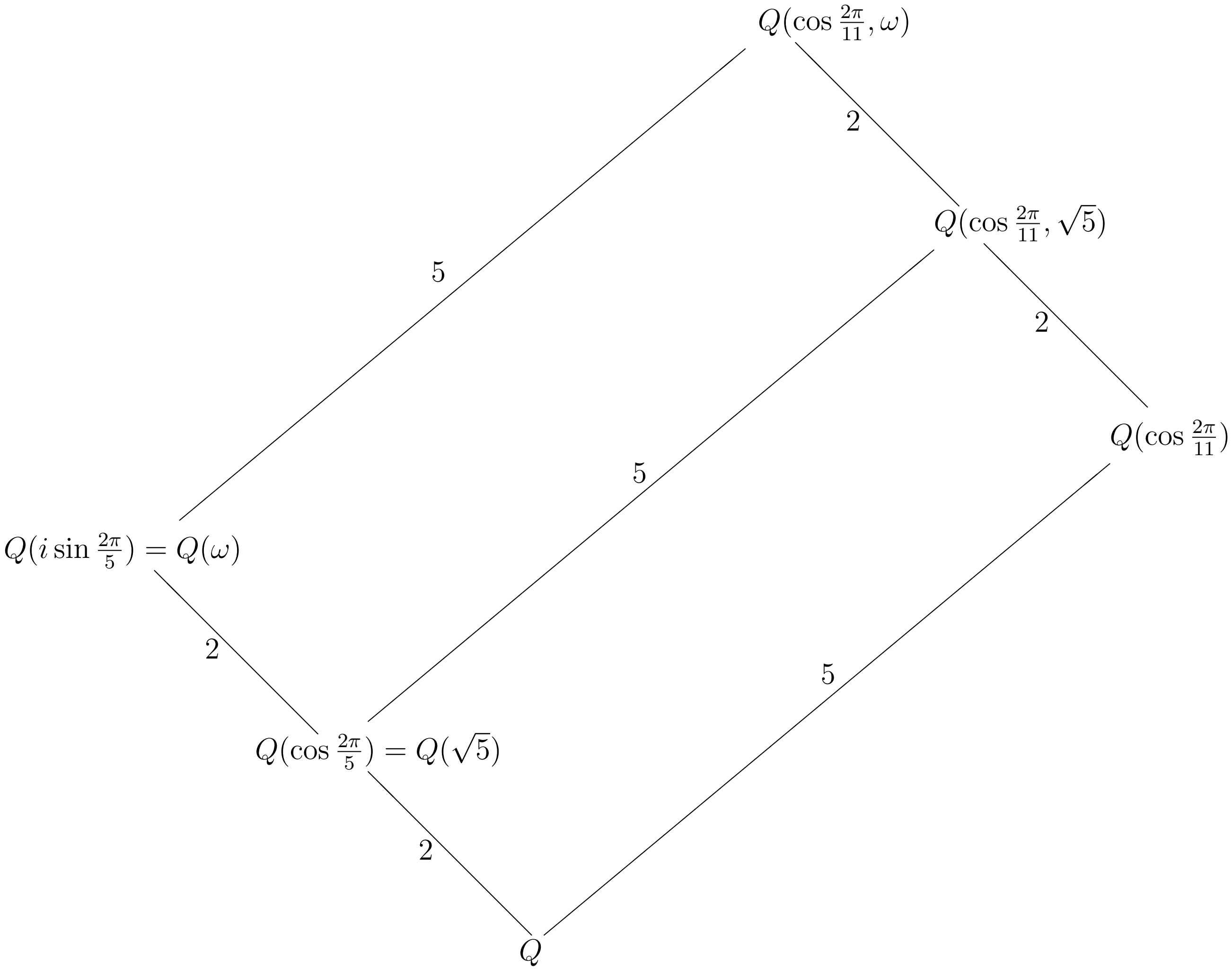
$$\begin{aligned}
V_1 &= a + b\omega + d\omega^2 + c\omega^3 + e\omega^4 \\
V_2 &= a + b\omega^2 + d\omega^4 + c\omega + e\omega^3 \\
V_3 &= a + b\omega^3 + d\omega + c\omega^4 + e\omega^2 \\
V_4 &= a + b\omega^4 + d\omega^3 + c\omega^2 + e\omega
\end{aligned}$$

We note that $V_i$, for $1 \le i \le 4$ lie in the field $K = Q(\cos\frac{2\pi}{11}, \omega)$ which is of degree 20 over $Q$. It is easily seen that

$$\begin{aligned}
a &= \tfrac{1}{5}\left(1 + V_1 + V_2 + V_3 + V_4\right) \\
b &= \tfrac{1}{5}\left(1 + \omega^4 V_1 + \omega^3 V_2 + \omega^2 V_3 + \omega V_4\right) \\
c &= \tfrac{1}{5}\left(1 + \omega^3 V_1 + \omega V_2 + \omega^4 V_3 + \omega^2 V_4\right) \\
d &= \tfrac{1}{5}\left(1 + \omega^2 V_1 + \omega^4 V_2 + \omega V_3 + \omega^3 V_4\right) \\
e &= \tfrac{1}{5}\left(1 + \omega V_1 + \omega^2 V_2 + \omega^3 V_3 + \omega^4 V_4\right)
\end{aligned}$$

Since we already have radical expression for $\omega$, it is enough to give a radical expression for $V_i's$ for $1 \le i \le 4$. This can be done by showing that the fifth powers of $V_i's$ lie in the field $Q(\omega)$ and we shall have the radical expressions for the roots of (4) as soon as we have an explicit expression for the fifth powers of $V_i's$ as members of $Q(\omega)$.

Consider the following diagram of the subfields of $K = Q(\cos\frac{2\pi}{11}, \omega)$



Evidently the Galois group of $G = Aut(K/Q)$ is cyclic of order 20. Let $\sigma$ denotes the automorphism of $K$ which fixes $\omega$ but permutes the five roots of (4) cyclically as:

$$a \to b \to d \to c \to e \to a$$

In other words $\sigma$ is the cyclic permutation $(a, b, d, c, e)$. Clearly the fixed field of the subgroup $\langle\sigma\rangle$ of $G = Aut(K/Q)$ is $Q(\omega)$. In fact if we let $\eta$ be the automorphism of $K$ which maps $\omega$ to $\omega^2$ and fixes $a$ (and therefore the subfield $Q(\cos\frac{2\pi}{11})$), then $G = Aut(K/Q) = \langle\sigma\eta\rangle$. The Table I below gives the effect of $\sigma$ and $\eta$ on $a, b, c, d, e$ and $\omega$ and the Table II gives the Galois correspondence between the subfields of $K$ and the subgroups of $G = Aut(K/Q)$.

Table I

| $x$ | $\sigma(x)$ | $\eta(x)$ | $\sigma\eta(x)$ |
|---|---|---|---|
| $\omega$ | $\omega$ | $\omega^2$ | $\omega^2$ |
| $a$ | $b$ | $a$ | $b$ |
| $b$ | $d$ | $b$ | $d$ |
| $d$ | $c$ | $d$ | $c$ |
| $c$ | $e$ | $c$ | $e$ |
| $e$ | $a$ | $e$ | $a$ |

Table II

| Subgroup | Fixed Subfield |
|---|---|
| $G = \langle \sigma\eta \rangle$ | $Q$ |
| $\langle \sigma\eta^2 \rangle$ | $Q(\cos\frac{2\pi}{5}) = Q(\sqrt{5})$ |
| $\langle \sigma \rangle$ | $Q(i\sin\frac{2\pi}{5}) = Q(\omega)$ |
| $\langle \eta \rangle$ | $Q(\cos\frac{2\pi}{11}) = Q(a)$ |
| $\langle \eta^2 \rangle$ | $(\cos\frac{2\pi}{11}, \cos\frac{2\pi}{5}) = Q(a, \sqrt{5})$ |
| $\{e\}$ | $Q(\cos\frac{2\pi}{11}, \omega)$ |

Now as $V_1 = a + b\omega + d\omega^2 + c\omega^3 + e\omega^4$, we have

$$\sigma(V_1) = b + d\omega + c\omega^2 + e\omega^3 + a\omega^4 = \omega^4(a + b\omega + d\omega^2 + c\omega^3 + e\omega^4) = \omega^4 V_1$$

Therefore $\sigma(V_1^5) = V_1^5$ and so the fixed subfield of $\langle\sigma\rangle$ is $Q(\omega)$. In other words $V_1^5 \in Q(\omega)$. Likewise $V_2^5, V_3^5, V_4^5 \in Q(\omega)$.

Although we know that $V_1^5 \in Q(\omega)$ but nevertheless it is quite cumbersome to find an actual expression for it as an element of $Q(\omega)$. It is hard to imagine how so very patiently Vandermonde may have carried out this computation. While doing calculation by hand it would be unwise to write down the $5^5 = 3125$ terms of $V_1^5 = (a + b\omega + d\omega^2 + c\omega^3 + e\omega^4)^5$ and then attempt to simplify these using the 15 relations from Vandermonde's paper indicated earlier (and the fact that $\omega^5 = 1$ and $a + b + c + d + e = 1$). The right approach would be to find the values of $V_1^2$, $V_1^3$, $V_1^4$ and $V_1^5$ in succession and simplifying each (i.e. expressing it as a linear polynomial in $a, b, c, d$, and $e$ with coefficients in $Q(\omega)$) before attempting to find the next higher power. Undoubtedly, Vandermonde followed this approach although he neither so claims and nor does he give the expressions for $V_1^2$, $V_1^3$, $V_1^4$ and $V_1^5$ in his paper. The fact that Vandermonde's end result is slightly incorrect does not in any way diminishes our appreciation for his patiently carrying out enormous calculation by hand.

Since the automorphism $\eta$ of $K$ permutes $V_i's$ cyclically as $V_1 \to V_2 \to V_4 \to V_3 \to V_1$, we only need to compute $V_1^5$. We shall not deprive the readers the joy of calculating $V_1^5$ by hand. Here we shall content ourselves by giving the explicit expressions for $V_1^2$, $V_1^3$, $V_1^4$ and $V_1^5$. We have checked their correctness with the help of **MATHEMATICA**. For the purposes of calculations we shall find it more convenient to write each power of $V_i$ as a polynomial in $\omega$ with coefficients which are linear in $a, b, c, d$ and $e$.

$$\begin{aligned}
V_1 &= a + b\omega + d\omega^2 + c\omega^3 + e\omega^4 \\
V_1^2 &= (b - 2d + 2e) + (-2a + 2d + e)\omega + (2a - 2c + d)\omega^2 + (a - 2b + 2c)\omega^3 + (2b + c - 2e)\omega^4 \\
V_1^3 &= (2 - 8a - 2b + 2c - 5d + 4e) + (2 + 2a + 4b - 2c - 8d - 5e)\omega + \\
&\quad (2 - 2a - 5b + 4c + 2d - 8e)\omega^2 + (2 + 4a - 8b - 5c - 2d + 2e)\omega^3 + \\
&\quad (2 - 5a + 2b - 8c + 4d - 2e)\omega^4 \\
V_1^4 &= (-16 + 34a + 28b + 18c - 7d + 8e) + (-16 + 28a - 7b + 8c + 18d + 34e)\omega + \\
&\quad (-16 - 7a + 18b + 34c + 8d + 28e)\omega^2 + (-16 + 18a + 8b + 28c + 34d - 7e))\omega^3 + \\
&\quad (-16 + 8a + 34b - 7c + 28d + 18e)\omega^4 \\
V_1^5 &= 196 + 130\omega - 255\omega^2 + 20\omega^3 - 90\omega^4
\end{aligned}$$

In a like manner (or preferably by simply applying the automorphism $\eta$) we obtain similar expressions for $V_2^5, V_3^5$ and $V_4^5$. These expressions are:

$$V_2^5 = 196 + 20\omega + 130\omega^2 - 90\omega^3 - 255\omega^4$$

$$V_3^5 = 196 - 255\omega - 90\omega^2 + 130\omega^3 + 20\omega^4$$

$$V_4^5 = 196 - 90\omega + 20\omega^2 - 255\omega^3 + 130\omega^4$$

By substituting the values of $\omega^i$ for $1 \leq i \leq 4$ in radical form we get the following radical expressions for $V_j^5$, for $1 \leq j \leq 4$.

$$V_1^5 = \frac{11}{4}\left(89 + 25\sqrt{5}\right) + \frac{11}{4}\left(20\sqrt{10 + 2\sqrt{5}} - 25\sqrt{10 - 2\sqrt{5}}\right) i$$

$$V_2^5 = \frac{11}{4}\left(89 - 25\sqrt{5}\right) + \frac{11}{4}\left(25\sqrt{10 + 2\sqrt{5}} + 20\sqrt{10 - 2\sqrt{5}}\right) i$$

$$V_3^5 = \frac{11}{4}\left(89 - 25\sqrt{5}\right) - \frac{11}{4}\left(25\sqrt{10 + 2\sqrt{5}} + 20\sqrt{10 - 2\sqrt{5}}\right) i$$

$$V_4^5 = \frac{11}{4}\left(89 + 25\sqrt{5}\right) - \frac{11}{4}\left(20\sqrt{10 + 2\sqrt{5}} - 25\sqrt{10 - 2\sqrt{5}}\right) i$$

So we conclude that the following correction need to applied to the values of $\Delta^{'}, \Delta^{''}, \Delta^{'''}, \Delta^{iv}$ given by Vandermonde.

Vandermonde's $\Delta^{'^5}$ should be replace by our $V_1^5$.
Vandermonde's $\Delta^{''^5}$ should be replace by our $V_4^5$.
Vandermonde's $\Delta^{'''^5}$ should be replace by our $V_3^5$.
Vandermonde's $\Delta^{iv^5}$ should be replace by our $V_2^5$.

We observe that $V_1 \cdot V_4 = V_2 \cdot V_3 = 11$. With appropriate choice of fifth roots of $V_1^5, V_2^5, V_3^5$ and $V_4^5$ we can find the radical expressions for the roots of the equation (4). However to check the correctness of our expressions for $V_1^5, V_2^5, V_3^5$ and $V_4^5$ we proceed as follows.

Let $\theta_1$ and $\theta_2$ be defined as follows:

$$\theta_1 = arctan\left(\frac{20\sqrt{10+2\sqrt{5}}-25\sqrt{10-2\sqrt{5}}}{89+25\sqrt{5}}\right) \approx 6.8107268^o$$

$$\theta_2 = arctan\left(\frac{25\sqrt{10+2\sqrt{5}}+20\sqrt{10-2\sqrt{5}}}{89-25\sqrt{5}}\right) \approx 76.890834^o$$

then

$$a = -2\cos\frac{2\pi}{11} = \frac{1}{5}\left[1+2\sqrt{11}\left\{\cos\frac{\theta_1+6\pi}{5}+\cos\frac{\theta_2+6\pi}{5}\right\}\right]$$

$$b = -2\cos\frac{4\pi}{11} = \frac{1}{5}\left[1+2\sqrt{11}\left\{\cos\frac{\theta_1+4\pi}{5}+\cos\frac{\theta_2+2\pi}{5}\right\}\right]$$

$$c = -2\cos\frac{6\pi}{11} = \frac{1}{5}\left[1+2\sqrt{11}\left\{\cos\frac{\theta_1}{5}+\cos\frac{\theta_2+4\pi}{5}\right\}\right]$$

$$d = -2\cos\frac{8\pi}{11} = \frac{1}{5}\left[1+2\sqrt{11}\left\{\cos\frac{\theta_1+2\pi}{5}+\cos\frac{\theta_2+8\pi}{5}\right\}\right]$$

$$e = -2\cos\frac{10\pi}{11} = \frac{1}{5}\left[1+2\sqrt{11}\left\{\cos\frac{\theta_1+8\pi}{5}+\cos\frac{\theta_2}{5}\right\}\right]$$

The correctness of these expression can be easily checked with the help of a calculator. Finally for the curious we have the following explicit radical expression for $\cos\frac{2\pi}{11}$.

$$\cos\frac{2\pi}{11} = -\frac{1}{2}a = -\frac{1}{10}\left(1+V_1+V_2+V_3+V_4\right) =$$

$$-\frac{1}{10}\left[1+\left\{\frac{11}{4}\left(89+25\sqrt{5}\right)+\frac{11}{4}\left(20\sqrt{10+2\sqrt{5}}-25\sqrt{10-2\sqrt{5}}\right)i\right\}^{\frac{1}{5}}+\right.$$

$$\left\{\frac{11}{4}\left(89-25\sqrt{5}\right)+\frac{11}{4}\left(25\sqrt{10+2\sqrt{5}}+20\sqrt{10-2\sqrt{5}}\right)i\right\}^{\frac{1}{5}}+$$

$$\left\{\frac{11}{4}\left(89-25\sqrt{5}\right)-\frac{11}{4}\left(25\sqrt{10+2\sqrt{5}}+20\sqrt{10-2\sqrt{5}}\right)i\right\}^{\frac{1}{5}}+$$

$$\left.\left\{\frac{11}{4}\left(89+25\sqrt{5}\right)-\frac{11}{4}\left(20\sqrt{10+2\sqrt{5}}-25\sqrt{10-2\sqrt{5}}\right)i\right\}^{\frac{1}{5}}\right]$$

This indeed gives correct value of $\cos\frac{2\pi}{11}$ with a proper choice of the fifth roots of unity in the above expression.

**Acknowledgment:** We would like to express our thanks and appreciation to an anonymous referee for bringing to our attention the Lagrange's comments about the work of Vandermonde that we have quoted in this article.

## REFERENCES

1. H. M. Edwards, *Galois Theory*, Springer-Verlag, 1984.
2. Joseph Lagrange, *Oeuvres*, p 354-5 (Gauthier-Villars, Paris, 1869), Notes sur la thérie des équations, Note XIV (*Où l'on donne la résolution générale des équations à deux termes*).
3. Robin Rider Hamburg, *The Theory of Equations in the* $18^{th}$ *century. The work of Joseph Lagrange*, Archive for the History of Exact Sciences, vol 16, 1976-77, 17-36.
4. Jean-Pierre Tignol, *Galois' Theory of Algebraic Equations*, World Scientific, 2001.
5. A. T. Vandermonde, *Mémoire sur la résolution des équation*, Histoire de l'Acad. Royale des Sciences (avec les mémoires de Math. & Phys. pour la même année, tirés des registres de cette Acad.) 1771, 365-416.