

# Continuity in Semi-Metric Spaces

Gregory A. Wisloski\*

## Abstract

We investigate the continuity of any semi-metric on the space  $\Psi$ . Next, we show that a regular space with a continuous semi-metric is completely regular. Finally, using conditional countable compactness and the 3-link property, we get conditions under which a regular semi-metrizable space is compact metric.

## 1 The space $\Psi$

The construction of the space  $\Psi$  used in this paper is from Gillman and Jerison [1]. We define the space  $\Psi = \mathbb{N} \cup \mathcal{S}(\mathbb{N})$ , where  $\mathcal{S}(\mathbb{N})$  is an uncountable collection of almost disjoint subsets of  $\mathbb{N}$ . At this point, it is prudent to define the topology of  $\Psi$  in terms of local bases, as this will give us a handle as to what's really happening with this space.

- For each  $x \in \mathbb{N}$ ,  $\{\{x\}\}$  is a local base.
- For each  $s \in \mathcal{S}(\mathbb{N})$ , consider  $s$  to be a strictly increasing sequence  $\{s_i\}$ , with each  $s_i \in \mathbb{N}$ . Define  $g_n(s) = \{s\} \cup \{s_i : i \geq n\}$ . A local base for  $s$  is  $\{g_n(s) : n \in \mathbb{N}\}$ .

Note that it will be convenient to use the notation  $g_n(s)$ , as defined above, for all  $s \in \mathcal{S}(\mathbb{N})$ , and all  $n \in \mathbb{N}$ . The space  $\Psi$  satisfies many important topological properties. It is Hausdorff, first countable, separable, locally compact, developable, and most importantly for our discussions, semi-metrizable.

Any development for  $\Psi$  induces a natural semi-metric on the space. For the sake of the current discussion, we introduce a new semi-metric,  $\delta$ , on  $\Psi$  as follows:

---

\*Indiana University of Pennsylvania, Department of Mathematics, Indiana, PA 15705  
wisloski@iup.edu

- If  $p \in \mathcal{S}(\mathbb{N})$  and  $n \in p$ , then  $\delta(n, p) = \frac{1}{n}$
- If  $p \in \mathcal{S}(\mathbb{N})$ ,  $n \in \mathbb{N}$ , and  $n \notin p$ , then  $\delta(n, p) = 1 + \frac{1}{n}$
- If  $m, n \in p \in \mathcal{S}(\mathbb{N})$ , then  $\delta(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right|$
- If  $p, q \in \mathcal{S}(\mathbb{N})$ , then  $\delta(p, q) = 1$  (Also for  $p \in \mathcal{S}(\mathbb{N})$ ,  $\delta(p, p) = 0$ )
- If  $m, n \in \mathbb{N}$  and  $\{m, n\} \not\subseteq p$  for any  $p \in \mathcal{S}(\mathbb{N})$ , then  $\delta(m, n) = 1 + \frac{1}{m} + \frac{1}{n}$

Neither the natural semi-metric or the one defined above are continuous. So there is no hope that either semi-metric is a metric on the space. Actually, it turns out that the space  $\Psi$  can never have a (non-trivial) continuous semi-metric defined on it.

**Theorem 1.1** *The space  $\Psi$  cannot have a continuous semi-metric.*

**Proof:** Suppose  $\Psi$  has a continuous semi-metric  $d$ . Define  $U_j(x) = \{y \in \Psi : d(x, y) < j\}$ . Assume, without loss of generality, that  $U_{\frac{1}{n}}(s) \subseteq g_1(s)$  for each  $s \in \mathcal{S}(\mathbb{N})$ . Since  $\mathcal{S}(\mathbb{N})$  is uncountable, there exists  $x_1 \in \mathbb{N}$  such that  $x_1 \in U_1(s)$  for uncountably many  $s \in \mathcal{S}(\mathbb{N})$ , so that  $U_1(x_1) \cap \mathcal{S}(\mathbb{N})$  is uncountable. Similarly, pick  $x_2 \in \mathbb{N}$  such that  $U_{\frac{1}{2}}(x_2) \cap (U_1(x_1) \cap \mathcal{S}(\mathbb{N}))$  is uncountable. By induction, for each  $n$  we can pick  $x_n$  so that  $\bigcap \{U_{\frac{1}{i}}(x_i) \cap \mathcal{S}(\mathbb{N}) : i \geq n\}$  is uncountable. Observe that either the sequence  $\{x_n\}$  belongs to  $\mathcal{S}(\mathbb{N})$ , or  $\{x_n : n \in \mathbb{N}\}$  has infinite intersection with some  $s \in \mathcal{S}(\mathbb{N})$ .  $\{x_n : x_n \in s \in \mathcal{S}(\mathbb{N})\}$  could have been the chosen sequence  $\{x_n\}$ . So without loss of generality, let  $p = \{x_n\} \in \mathcal{S}(\mathbb{N})$ , and  $U_{\frac{1}{n}}(p) = \{s_i : i \geq n\}$ .

Note that  $|U_{\frac{1}{n}}(x_n) \cap \mathcal{S}(\mathbb{N})| \geq \omega_1$ . Actually what we have is that there is an uncountable set of points of  $\mathcal{S}(\mathbb{N})$  (call it  $S_1$ ) that is contained in  $U_1(x_1)$ . Now inductively define a decreasing sequence  $\{S_n\}$  as follows:  $S_n \subset S_{n-1}$ ,  $S_n \subset U_{\frac{1}{n}}(x_n)$ ,  $|S_n| \geq \omega_1$  for all  $n \in \mathbb{N}$ .

Now for all  $n \in \mathbb{N}$ ,  $\overline{U_{\frac{1}{n}}(p)} \cap (S_n \setminus \{p\}) = \emptyset$ . This allows us to choose our sequence  $\{y_n\}$ , with  $\lim_{n \rightarrow \infty} (y_n, q) = 0$ ,  $q \in \mathcal{S}(\mathbb{N})$ . For each  $n \in \mathbb{N}$ , let  $y_n \in (\mathbb{N} \setminus U_{\frac{1}{n}}(p)) \cap U_{\frac{1}{n}}(x_n)$ .

By assumption, we have  $\lim_{n \rightarrow \infty} d(x_n, p) = 0$ ,  $\lim_{n \rightarrow \infty} d(y_n, q) = 0$ . By the construction of  $\{y_n\}$  we have  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ . But  $p \neq q$ , so  $\lim_{n \rightarrow \infty} d(p, q) > 0$ . So the semi-metric  $d$  cannot be continuous. ■

Although we do not have continuity in any semi-metric on  $\Psi$ , the semi-metric  $\delta$  is (conveniently) separately continuous.

**Theorem 1.2** *The space  $\Psi$  has a semi-metric on it that is separately continuous.*

**Proof:** Note that since  $\delta$  is a semi-metric, we only need to show that  $f(x) = \delta(p, x)$ ,  $p$  fixed in  $\Psi$ , is continuous. In each case, we assume that  $y_i \rightarrow x_0$ :

- **Case 1:** Fix  $p \in \mathbb{N}$ . If we have the point  $x_0 \in \mathbb{N}$ ,  $\{x_0\}$  is open. It follows then that no sequence  $\{y_i\}$  can converge to  $x_0$  except for the sequence where each term  $y_i$  is identically equal to  $x_0$ . Separate continuity immediately follows.
- **Case 2:** Let  $x_0 \in \mathcal{S}(\mathbb{N})$ . Since  $y_i \rightarrow x_0$  we have  $y_i \in x_0$  for  $i \geq m$ , for some  $m$ . It follows that  $\delta(x_0, y_i) = \frac{1}{\text{index of } y_i \in x_0}$ . Now let  $p \in x_0$ ,  $p = s_k$  for some index  $k$ , and  $y_i = s_{m_i}$ . Then  $\delta(p, y_i) = \frac{1}{k} - \frac{1}{m_i}$ , which converges to  $\frac{1}{k}$ . Note we also have that  $f(x_0) = \delta(p, x_0) = \frac{1}{k}$ . So, finally, we have  $f(y_i) = \delta(p, y_i)$ , which by above converges to  $\frac{1}{k}$ . Then we have  $f(y_i) \rightarrow f(x_0)$ .
- **Case 3:** Let  $p \in \mathcal{S}(\mathbb{N})$ ,  $p \neq x_0$ . Then  $\delta(p, y_i) = 1 + \frac{1}{m_i}$ , for some index  $m_i$ . Then we have  $f(y_i) = \delta(p, y_i)$  converging to 1. Also, since  $x_0 \in \mathcal{S}(\mathbb{N})$ , we have  $f(x_0) = \delta(p, x_0) = 1$ . Thus  $f(y_i) \rightarrow f(x_0)$ .
- **Case 4:** Now suppose that  $p \notin \mathcal{S}(\mathbb{N})$ , but  $p \in \mathbb{N}$ . Then  $f(y_i) = \delta(p, y_i) = 1 + \frac{1}{y_i} + \frac{1}{p}$ , which converges to  $1 + \frac{1}{p}$ . We also have  $f(x_0) = \delta(p, x_0) = 1 + \frac{1}{p}$ . Again we have  $f(y_i) \rightarrow f(x_0)$ . ■

## 2 Continuity in semi-metric spaces

The motivation for the following result is from Urysohn's Lemma. It is born out of the fact that Urysohn's Lemma does not follow for regular spaces, hence the separation axiom of complete regularity  $(T_{3\frac{1}{2}})$  is needed between regularity and normality.

**Theorem 2.1** *Every regular space  $X$  with a continuous semi-metric is  $T_{3\frac{1}{2}}$ .*

**Proof:** Let  $d$  be a continuous semi-metric. Since  $X$  is regular, for each point  $p$  and closed set  $M$ , there are open sets  $U$  and  $V$  such that  $p \in U$ ,  $M \subset V$ , and  $U \cap V = \emptyset$ . This guarantees that the distance between the point  $p$  and closed set  $M$  is non-zero. Since  $d$  is continuous, we want to fix the distance from our point  $p$  to any element of the closed set  $M$ . (i.e.,  $d(p, M) = D$ , with  $D > 0$ ) Define the continuous function  $f : X \rightarrow [0, 1]$  as  $f(x) = \frac{\min\{d(p, x), D\}}{D}$ . Since  $d$  is a semi-metric, we have  $f(p) = \frac{d(p, p)}{D} = \frac{0}{D} = 0$ . Also, for any  $x \in M$ ,  $f(x) = \frac{d(p, x)}{D} = \frac{D}{D} = 1$ . Thus,  $f(M) = 1$ . Then the space  $X$  is completely regular, or  $T_{3\frac{1}{2}}$ . ■

### 3 Conditional countable compactness and the 3-link property

The space  $X$  is said to be **conditionally countably compact** if  $X$  has a dense subspace  $D$  such that for every infinite subset  $A \subseteq D$ , there exists a limit point of  $A$  in  $X$ .

**Proposition 3.1** *The space  $\Psi$  is conditionally countably compact.*

**Proof:** From the definition of  $\Psi$ , we have an infinite maximal family  $\mathcal{C}$  of infinite subsets of  $\mathbb{N}$  such that the intersection of any two of them is finite. We define  $\mathcal{S}(\mathbb{N}) = \{\omega_C\}_{C \in \mathcal{C}}$  as our set of “ideal” points (this is a different method of visualizing what is happening in  $\Psi$ ) and  $\Psi = \mathbb{N} \cup \mathcal{S}(\mathbb{N})$ . Choose  $K$  as an infinite subset of  $\mathbb{N}$ . For some  $C \in \mathcal{C}$ , if  $K \cap C$  is infinite, then  $K \cap C$  has one of the ideal points  $\omega_C$  as its sequential limit. If not, then this is a contradiction to the maximality of  $\mathcal{C}$ . So in either case, we have that  $\Psi$  is conditionally countably compact. ■

Note that although the space  $\Psi$  is conditionally countably compact, it is not countably compact.

A development  $\{G_n\}$  for the space  $X$  is said to satisfy the **3-link property** if for any distinct points  $p, q$  in  $X$ , there is an integer  $n$  such that no member of  $G_n$  intersects both  $\text{st}(x, G_n)$  and  $\text{st}(y, G_n)$ .

Now that we have the machinery in place, we can characterize compact metric spaces in terms of both the 3-link property and conditional countable compactness.

**Theorem 3.2** *If  $X$  is a regular semi-metrizable space that satisfies the 3-link property and is conditionally countably compact, then  $X$  is compact metric.*

**Proof:** Define  $\hat{X} = X \cup X'$ , where  $X'$  is the set of all limit points of  $X$ . Since  $X$  is conditionally countably compact, every subset of  $X$  has a limit point in  $\hat{X}$ . Assume  $X$  is not a compact metric space, so there is a sequence  $\{y_i\} \subset \hat{X} \setminus X$  with no limit point. Now, consider a sequence of points  $\{x_{i_j}\}$  in  $X$  such that  $x_{i_j} \rightarrow p$ , for some  $p \in \hat{X}$ . Note that there exists an open set  $V$ , with  $p \in V \subset \bar{V} \subset \hat{X} \setminus \{y_i : i \in \mathbb{N}\}$ . Since  $X$  is a regular semi-metric space, there is a semi-metric  $d$  with  $d(x_{i_j}, y_i) < \frac{1}{j}$ . Pick  $x_{i_{j_i}} \in \hat{X} \setminus \{y_i : i \in \mathbb{N}\}$  with  $j_i > i$ . Then we have the point  $p \in \hat{X}$  with  $x_{i_{j_i}} \rightarrow p$ . If there's just this one such  $p$ , then this contradicts the conditional countable compactness of  $X$ . If we have two such points  $p$  and  $p'$ , with  $x_{i_{j_i}} \rightarrow p$  and  $x_{i_{k_i}} \rightarrow p'$ , then we have  $d(x_{i_{j_i}}, y_i) < \frac{1}{i}$  and  $d(x_{i_{k_i}}, y_i) < \frac{1}{i}$ . This tells us that  $x_{i_{j_i}} \in N_{\frac{1}{i}}(y_i) \cap N_{\frac{1}{i}}(p)$ , and  $x_{i_{k_i}} \in N_{\frac{1}{i}}(y_i) \cap N_{\frac{1}{i}}(p')$ . This then violates the 3-link property. ■

**Corollary 3.3**  *$\Psi$  does not have the 3-link property.*

If  $\Psi$  did satisfy the 3-link property, since it is semi-metrizable and conditionally countably compact, it would be compact hence metrizable.

## References

- [1] Gillman, L., Jerison, M., Rings of Continuous Functions, Van Nostrand, Princeton, New Jersey, 1960