The What, How, and Why of Pi

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Abstract

As we learn more mathematics, our implicit homework assignment in life is to go back to facts that we simply accepted as truth and think about why they are true. The homework assignment addressed in this paper is to think about π and its presence in all of those circle formulas. A simple acquaintance with limits will suffice.

1 The Problem

We define π to be the ratio of the circumference C of *any* circle to its diameter d (i.e. twice its radius r). That is,

$$\pi = \frac{C}{d} = \frac{C}{2r}.$$

However, this brings up the issue of whether or not π is truly well-defined. That is,

> (1) Does the diameter (radius) of the circle considered truly not affect the value of π?

Once we answer this question in the affirmative, it follows immediately that we have the expected formula for the circumference of a circle

$$C = \pi d = 2\pi r.$$

However, this leaves us with a second main question.

> (2) Based upon our definition of π, why is the area A of the circle then determined by the expected formula $A = \pi r^2$?

2 Regular n-gons

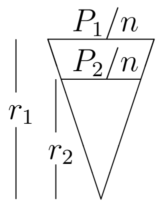
Before we answer our questions about circles, we first consider analogous questions regarding regular n-gons. To this end, define the radius (also known as the apothem) of a regular n-gon as the distance from the center of the n-gon to the center of one of its sides. Consequently, by the diameter of an n-gon, we shall mean twice its radius. Then, define π_n to be the ratio of the perimeter (circumference) C of *any* regular n-gon to its diameter $d = 2r$. That is,

$$\pi_n = \frac{C}{d} = \frac{C}{2r}.$$

As with the analogous problem for circles, this brings up the issue of whether or not π_n is truly well-defined. That is,

> (1.n) Does the diameter (radius) of the regular n-gon considered truly not affect the value of π_n?

This time we can easily answer yes to this question. It follows by similar triangles. If, for $i = 1, 2$, we let P_i be the perimeter of a regular n-gon with radius r_i, then P_1/n is to r_1 as P_2/n is to r_2.



Hence,

$$\frac{P_1}{2r_1} = \frac{P_2}{2r_2},$$

and π_n is seen to be well-defined.

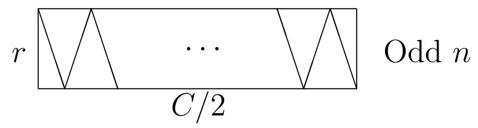
Of course, it now follows that we have a simple formula for the circumference of a regular n-gon

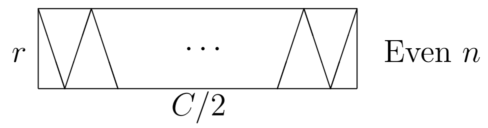
$$C = \pi_n d = 2\pi_n r.$$

However, this leaves us with a question about area.

> (2.n) Based upon our definition of π_n, what is the formula for the area of a regular n-gon?

A regular n-gon can be taken apart and reconstructed as a rectangle.





The area A of a regular n-gon is now seen to be given by

$$A = \frac{C}{2}r = \frac{2\pi_n r}{2}r = \pi_n r^2.$$

Hence, for regular n-gons, we have the analog of our desired result for circles.

3 Back to Circles

We can use our results for regular n-gons to answer our questions about circles. Of course, we are going to pass to limits, but we need to make this precise.

For a fixed circle of radius r, consider both a circumscribed and an inscribed regular n-gon. Note that the circumscribed n-gon will have the same radius as the circle. Let C_n be the perimeter of the circumscribed n-gon. For the inscribed n-gon, let r_n and I_n be the radius and perimeter, respectively.

It is clear that

$$I_n \leq C \leq C_n$$

and

$$r_n \to r \quad \text{as } n \to \infty.$$

By our results for regular n-gons,

$$\frac{C_n}{2r} = \pi_n = \frac{I_n}{2r_n}.$$

So,

$$C_n - I_n = I_n \left(\frac{r - r_n}{r_n} \right) \to 0.$$

Since I_n is bounded above by C, it follows that

$$C_n, I_n \to C.$$

Therefore,

$$\pi_n = \frac{C_n}{2r} \to \frac{C}{2r} = \pi.$$

Since the terms in the sequence

$$\{\pi_n\} = \left\{\frac{C_n}{2r}\right\}$$

are independent of the value of r, the limit of this sequence must also be independent of the value of r. Hence, this settles question (1) and tells us that π is indeed well-defined. Moreover, $C = 2\pi r$. So it remains to establish the formula for the area A of a circle as addressed in question (2).

By using our formula for the areas of the inscribed and circumscribed regular n-gons, we get

$$\pi_n r_n^2 \leq A \leq \pi_n r^2.$$

Since

$$\pi_n \to \pi \quad \text{and} \quad r_n \to r \quad \text{as} \quad n \to \infty,$$

it follows that

$$A = \pi r^2.$$

Therefore, question (2) has also been answered.

4 Other Problems

Having successfully completed the homework assignment of thinking about π, we ought to consider similar problems that should be addressed. One simple example is the problem of proving the Pythagorean Theorem $a^2 + b^2 = c^2$. There are many simple proofs of this based upon standard pictures which can be carved up and reconstructed in multiple ways. Another problem relates to a standard calculus example. If a farmer has a fixed length L of fence with which to enclose a pen, then what is the maximum area that can be enclosed? With the further restriction that the pen is a rectangle we get the standard calculus max-min problem whose solution is the area of the square of perimeter L. However, in general, the answer is the area of a circle of circumference L. Think about how to prove this.

5 References?

The point of this article is that, with just a bit of mathematical training, one ought to be able to recreate the arguments included here on one's own. Indeed, this is what the author has done. However, subsequent to this presentation, it has been suggested that the arguments might be the same as those used by Archimedes to answer the very same questions addressed here. This is likely so.