Bounding the Density of Abundant Numbers

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The notion of a perfect number, one that is the sum of its proper divisors, is over 2000 years old. Almost as old are the notions of abundant and deficient numbers, those whose divisor sums exceed or fall short of the numbers themselves. A look at the first 50 natural numbers shows two perfect numbers (6 and 28), nine abundant numbers (12, 18, 20, 24, 30, 36, 40, 42, and 48), and thus thirty-nine deficient numbers.

Perfect numbers are known to be quite rare; Euclid and Euler (two thousand years apart) established that an even number n is perfect if any only if $n = 2^{k-1}(2^k - 1)$, where $2^k - 1$ is prime. Such primes are called Mersenne primes; as of May 2003 there are 39 known to exist, with the largest over four million digits long. The existence of an odd perfect number is still unknown, though if one does exist, it must be greater than 10^{300}. Here we ask a simpler question: how common, or dense, are the abundant integers? To attempt to answer this, we look at the non-deficient integers - the perfect and abundant numbers lumped together.

$\sigma(n)$ is defined as the sum of all positive divisors of n, including n itself. Thus n is abundant, perfect, or deficient, respectively, as $\sigma(n)$ is greater than, equal to, or less than $2n$. n is nondeficient if $\sigma(n) \geq 2n$; that is, if it is abundant or perfect.

Let

$$A(x) = \lim_{n\to\infty} \frac{\#\{1 \leq i \leq n | \sigma i \geq xi\}}{n},$$

the density of integers for which $\sigma(n)$ exceeds xn for some real number x. Davenport (in 1933, [3]) and Erdős (in 1934, [7]) proved that the above limit exists and hence $A(x)$ is defined for any real x; in addition, $A(x)$ is continuous over the real numbers. Here we are concerned only with $A(2)$, the density of nondeficient integers. In 1933, Behrend [1] proved $.241 < A(2) < .314$; in 1972 Wall [10] refined this method to show $.2441 < A(2) < .2909$. In 1998, Deleglise [4], using further refinements and more computing power, showed $.2474 < A(2) < .2480$. In this paper, we find a lower bound for $A(2)$ using more elementary methods and the 1913 papers of L. E. Dickson as a starting point. Since our goal is an elementary result, we will give quick sketches of some facts that other papers have omitted.

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To begin, it is well known that if p is a prime, then $\sigma(p^n) = 1+p+p^2+\ldots+p^n$, and if m and n are relatively prime, then $\sigma(mn) = \sigma(m)\sigma(n)$.

Lemma 1 *Any multiple of a nondeficient number is also nondeficient.*

Proof: For any natural numbers m and n, $\sigma(mn) \geq m\sigma(n)$. To see this, let $1 = n_1, n_2, \ldots, n_r = n$ be the divisors of n. We have $m\sigma(n) = m + mn_2 + \ldots + mn$. All these are divisors of mn, so $\sigma(mn) \geq m\sigma(n)$. Now if n is nondeficient, $\sigma(mn) \geq m\sigma(n) \geq m(2n) = 2(mn)$.

Following L. E. Dickson, we define a **primitive nondeficient number (PNN)** to be a nondeficent number that is not a multiple of any other nondeficient number. Put another way, a PNN is a nondeficient number whose divisors are all deficient.

Dickson's 1913 papers [5, 6] list the first 121 primitive nondeficients, or all those less than 15000. The list begins $6, 20, 28, 70, 88, 104, 272, 304, \ldots$. By Lemma 1, we know every multiple of 6 is nondeficient, so we can write $A(2) \geq \frac{1}{6}$. Looking at the first two terms, we know $A(2) \geq \frac{1}{6} + \frac{1}{20} - \frac{1}{60}$. We can continue to use the inclusion/exclusion principle: for three terms we find $A(2) \geq \frac{1}{6} + \frac{1}{20} + \frac{1}{28} - \frac{1}{60} - \frac{1}{84} - \frac{1}{140} + \frac{1}{420} = \frac{92}{420}$. This very quickly becomes labor-intensive. We would like to write a recursive formula so that an estimate that accounts for the first n primitive nondeficients can be altered slightly to get the next estimate.

Toward this goal, define $\{a_i\}$ to be the sequence of PNNs, written in increasing order, $q_i = \mathrm{LCM}(a_1, \ldots, a_i)$, $< a_1, \ldots, a_i >=$ the set of all multiples of a_1 through a_i, and $p_i = |\{< a_1, \ldots, a_i > \cap [1, q_i]\}|$. Thus p_i is the number of integers between 1 and q_i (inclusive) that are multiples of one of the first i PNNs. The first few values of p_i and q_i can be computed by hand: $q_1 = 6$ and $p_1 = 1$ (the nondeficient number 6); $q_2 = \mathrm{LCM}(6, 20) = 60$ and $p_2 = 12$ (ten multiples of 6 and two "new" multiples of 20).

Since $a_j | q_i$, it should be evident that $ka_j + lq_i$ is nondeficient for any $j \leq i$; hence $\frac{p_i}{q_i}$ gives the density of multiples of the first i PNNs. Thus $\frac{p_i}{q_i} \leq \frac{p_{i+1}}{q_{i+1}} \leq A(2)$, and each $\frac{p_i}{q_i}$ gives a lower bound for $A(2)$.

The crucial question, then, in moving from $\frac{p_i}{q_i}$ to $\frac{p_{i+1}}{q_{i+1}}$ is "how many multiples of a_{i+1} are new?" That is, how many ka_{i+1} are not multiples of any a_j, with $j \leq i$?

Lemma 2 *Let $\{b_i\}$ be a sequence of natural numbers such that $b_{n+1} \notin < b_1, \ldots, b_n >$ for all n, and let $l_n = LCM(b_1, \ldots, b_n)$. Then*

$$\text{if } GCD(k, \frac{l_{n+1}}{b_{n+1}}) = 1, \text{ then } kb_{n+1} \notin < b_1, \ldots, b_n > .$$

Proof: Since $\mathrm{GCD}(k, \frac{l_{n+1}}{b_{n+1}}) = 1$ we know $kx + \frac{l_{n+1}}{b_{n+1}}y = 1$, or $kb_{n+1}x + l_{n+1}y = b_{n+1}$ for some integers x, y. Suppose that kb_{n+1} is a multiple of some earlier b_i; say $kb_{n+1} = jb_i$. Note that $l_{n+1} = mb_i$ for some m, so

$$b_{n+1} = kb_{n+1}x + l_{n+1}y = jb_ix + mb_iy = b_i(jx + my).$$

Thus $b_{n+1} \in < b_1, \ldots, b_n >$, which contradicts our assumption. So we must have $kb_{n+1} \notin < b_1, \ldots, b_n >$.

Lemma 3 *For any $n > 1$, $p_{i+1} \geq p_i \frac{q_{i+1}}{q_i} + \phi(\frac{q_{i+1}}{a_{i+1}})$*

Proof: p_{i+1} counts the number of multiples of $a_1, \ldots, a_{i+1}$ in the interval $[1, q_{i+1}]$. $p_i \frac{q_{i+1}}{q_i}$ is the number of multiples of $a_1, \ldots, a_i$. The set of PNNs, listed in increasing order, meet the hypotheses of Lemma 2, so a multiple ka_{i+1} of a_{i+1} is not a multiple of a previous a_j if $\mathrm{GCD}(k, \frac{q_{n+1}}{a_{n+1}}) = 1$. Thus there are at least $\phi(\frac{q_{i+1}}{a_{i+1}})$ of these "new" multiples (where $\phi(n)$, the Euler phi function, counts the number of natural numbers less than and relatively prime to n.)

Theorem 1 *Let a_i, p_i, and q_i be defined as above. Then $p_1 = 1$, $q_1 = 6$, and*

$$\frac{p_{i+1}}{q_{i+1}} \geq \frac{p_i}{q_i} + \frac{\phi(\frac{q_{i+1}}{a_{i+1}})}{q_{i+1}}.$$

Proof: Dividing the formula in Lemma 3 by q_{i+1} gives the desired result.

The only task remaining is to implement this. We use Dickson's list of 121 PNNs and Mathematica 3.0, making use of its built-in Euler ϕ function. To improve the approximation, we wrote a short C++ program to find the first 250 PNNs.

The resulting approximations are given below. Recall that a_i is the ith PNN and $\frac{p_i}{q_i}$ is the ith lower bound for $A(2)$.

i	a_i	$\frac{p_i}{q_i}$
1	6	.166667
2	20	.200000
3	28	.219048
5	88	.229004
10	464	.237233
25	1870	.241002
50	4510	.242425
100	9555	.243711
121	14824	.243927
142	20864	.244104
150	22912	.244161
200	33915	.244408
250	46436	.244552

It should be noted that $A(2)$ counts nondeficient rather than abundant numbers. To find lower bounds for the density of abundant integers, we need to remove the occasional perfect numbers. For example, $q_{121} \approx 5.48 \times 10^{60}$, so p_{121} counts 10 perfect numbers (see p. 220 of [2]). Thus to count only abundant numbers, decrease $\frac{p_{121}}{q_{121}}$ by $\frac{10}{q_{121}}$, or roughly 10^{-59}. Thus this method improves on Wall's 1972 result in the 142nd step. Deleglise's 1998 result, however, is still distant. Finally, note that as this method uses integers in each step, there is no accumulated roundoff error.

References

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