

Why Maximum Toughness is Tougher than Maximum Connectivity

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Abstract

Given any positive integers n and m , Frank Harary constructed a graph on n vertices and m edges whose connectivity cannot be exceeded by any graph on the same numbers of vertices and edges. These graphs are now known as Harary graphs and are said to be maximally connected. The corresponding problem in which connectivity is replaced by toughness turns out to be much harder. Sometimes the Harary graphs are maximally tough, but not always. The cases in which the maximum toughness problem is tougher than the maximum connectivity problem will be discussed.

1 Terminology

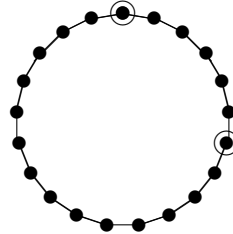
Let $\omega(G)$ denote the number of components in a graph G . The connectivity of G is

$$\kappa(G) = \min\{|S| : S \subseteq V, \omega(G - S) > 1\}.$$

The toughness of G is

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G - S)} : S \subseteq V, \omega(G - S) > 1\right\}.$$

Example 1.1. $\kappa(C_n) = 2$ and $\tau(C_n) = 1$.



An (n, m) -graph has $|V| = n$ and $|E| = m$. Maximum connectivity is defined by the function

$$C_n(m) = \max\{\kappa(G) : G \text{ is an } (n, m)\text{-graph}\}.$$

Maximum toughness is defined by the function

$$T_n(m) = \max\{\tau(G) : G \text{ is an } (n, m)\text{-graph}\}.$$

An (n, m) -graph G is maximally connected if $\kappa(G) = C_n(m)$ and maximally tough if $\tau(G) = T_n(m)$.

2 A Typical Example

Consider $n = 6$ and $m = 10$. That is, consider graphs on 6 vertices and 10 edges. What are $C_6(10)$ and $T_6(10)$? Figure 1 shows some graphs on 6 vertices and 10 edges that might be maximally connected or maximally tough. In fact, the maximum possible connectivity is 3. That is $C_6(10) = 3$.

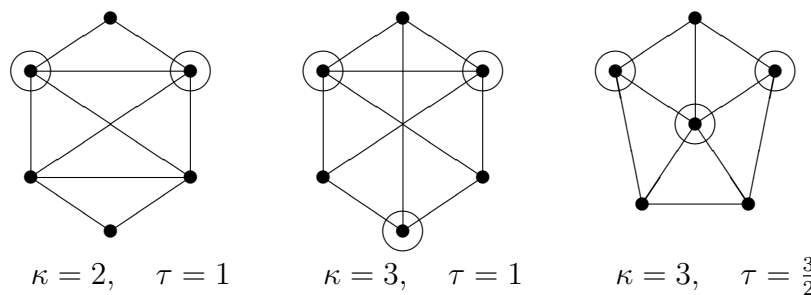


Figure 1: Some graphs on 6 vertices and 10 edges.

The maximum possible toughness is $\frac{3}{2}$. That is $T_6(10) = \frac{3}{2}$.

3 General Results

In most cases,

$$T_n(m) = \frac{C_n(m)}{2}$$

In general,

$$T_n(m) \leq \frac{C_n(m)}{2}$$

3.1 Harary Graphs $H(n, m)$

Harary computed $C_n(m)$.

Theorem 3.1 ([3]). For $n \geq 1$ and $0 \leq m \leq \frac{n(n-1)}{2}$,

$$C_n(m) = \begin{cases} 0 & \text{if } m \leq n - 2, \\ \lfloor \frac{2m}{n} \rfloor & \text{if } m \geq n - 1. \end{cases}$$

Harary constructed a family of graphs that are maximally connected. Figure 2 shows some examples that suggest the general construction. In

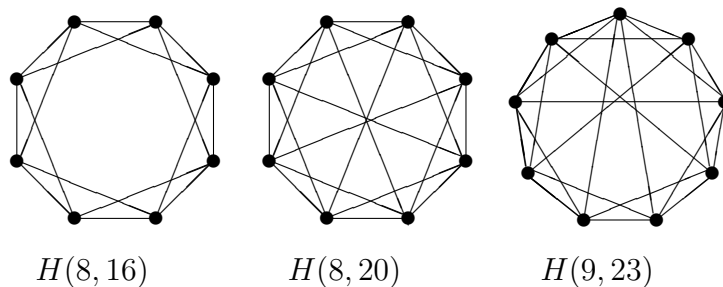


Figure 2: Examples of Harary Graphs.

general, Harary graphs are not maximally tough. However, many of them are.

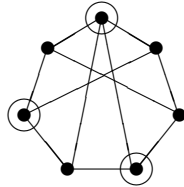
Theorem 3.2 ([1]). Let $n \geq 3$, $n \leq m \leq \frac{n(n-1)}{2}$, and $r = \lfloor \frac{2m}{n} \rfloor$. If

- (i) r is even, or
- (ii) $r \geq 2(2\lfloor \frac{n}{6} \rfloor + \lfloor \frac{n \bmod 6}{3} \rfloor)$

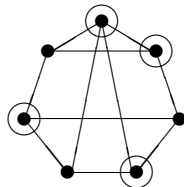
then $T_n(m) = \frac{r}{2}$.

3.2 Toughness is More Subtle

For $n = 7$ and $m = 11$, the Harary Graph $H(7, 11)$ is maximally connected (with connectivity 3)



and has toughness equal to 1. Whereas, the following $(7, 11)$ -graph



has toughness equal to $\frac{4}{3}$. So $H(7, 11)$ is maximally connected but not maximally tough. In fact, $T_7(11) = \frac{4}{3}$. Note that $T_7(11) < \frac{C_7(11)}{2}$. The central difficulty is that G being maximally connected does not imply that G is maximally tough. An open question is, does G being maximally tough imply that G is maximally connected?

4 A Bigger Question

Does the converse of the following theorem hold?

Theorem 4.1 ([5]). *If G is $K_{1,3}$ -free, then*

$$\tau(G) = \frac{\kappa(G)}{2}.$$

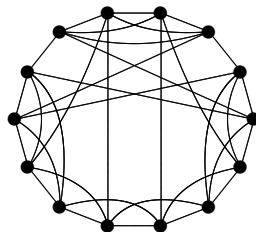
The following picture shows that subgraphs that must be avoided in the previous result.



Conjecture 4.2 ([2]). *For any $r \geq 3$, if G is r -regular and $\tau(G) = \frac{r}{2}$, then G is $K_{1,3}$ -free.*

Theorem 4.3 ([4]). *If G is cubic and $\tau(G) = \frac{3}{2}$, then G is $K_{1,3}$ -free.*

Here is an example of a 5-regular non- $K_{1,3}$ -free $\frac{5}{2}$ -tough graph that settles the conjecture in the negative.



References

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