Toughness of Graphs

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Abstract

The toughness of a graph is a measure of the graph's vulnerability to having vertices removed. For example, if the vertices of a graph represent homes, and the edges of the graph represent telephone lines connecting them, then the toughness measures how badly telephone communication can be broken down by relatively few lightening strikes. My primary goal is to give you a sense of the toughness of some important classes of graphs. Secondly, I will discuss my own results on the toughness of generalized Petersen graphs.

1 Graph Vulnerability

The graph parameter toughness is a particular example of a measure of vulnerability. That is, it reflects what happens when vertices of the graph are removed. Before we consider toughness, it will be useful to first consider a more basic measure of vulnerability.

1.1 Connectivity

The connectivity of a noncomplete graph $G = (V, E)$ is defined as

$$\kappa(G) = min\{|S| \ : S \subseteq V \text{ and } \omega(G - S) > 1\}, \quad (1.1)$$

where $\omega(G-S)$ is the number of components of the graph obtained from G by removing the vertices of S. For a complete graph K_n, we define $\kappa(K_n) \equiv \infty$.

We acquaint ourselves with connectivity by considering a few important classes of graphs. In each example, disconnecting vertices that achieve the connectivity are circled.

A path P_n on n vertices. For $n \geq 3$, $\kappa(P_n) = 1$.

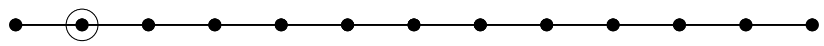

Figure 1: Achieving $\kappa(P_n)$

A star graph. For $n \geq 2$, $\kappa(K_{1,n}) = 1$.

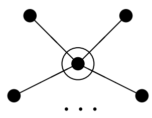

Figure 2: Achieving $\kappa(K_{1,n})$

Trees. Note that the two previous examples are both trees. In general, if $G = (V, E)$ is a tree with $|V| \geq 3$, then $\kappa(G) = 1$.

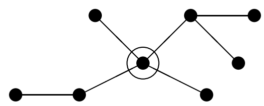

Figure 3: Achieving connectivity for trees.

A cycle on n vertices. For $n \geq 4$, $\kappa(C_n) = 2$.

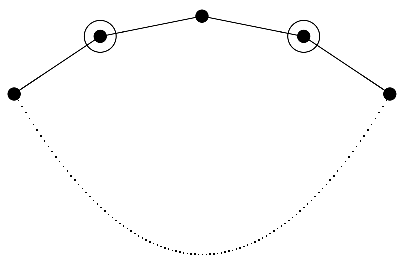

Figure 4: Achieving $\kappa(C_n)$

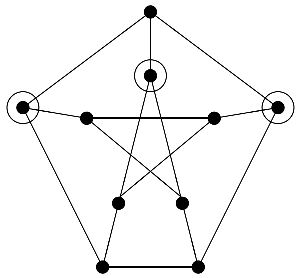

Figure 5: Achieving $\kappa(G(5,2))$

The Petersen Graph $G(5,2)$**.** The Petersen graph has $\kappa(G(5,2)) = 3$.

1.2 Toughness

The toughness of a non-complete graph $G = (V, E)$ is defined by Chvátal [3] as

$$t(G) = min\{\frac{|S|}{\omega(G-S)} \ : S \subseteq V \text{ and } \omega(G-S) > 1\}. \tag{1.2}$$

Recall that $\omega(G - S)$ is the number of components of the graph obtained from G by removing the vertices of S. For a complete graph K_n, we define $t(K_n) \equiv \infty$. Note that formally the definition of toughness is very similar to the definition of connectivity.

An understanding of toughness is best gained by considering several examples.

Trial and Error for $G(8,2)$**.** Determining the toughness of a graph usually involves some experimentation. The goal is to find a disconnecting set S that minimizes $\frac{|S|}{\omega(G-S)}$. Of course, any disconnecting set provides an upper bound.

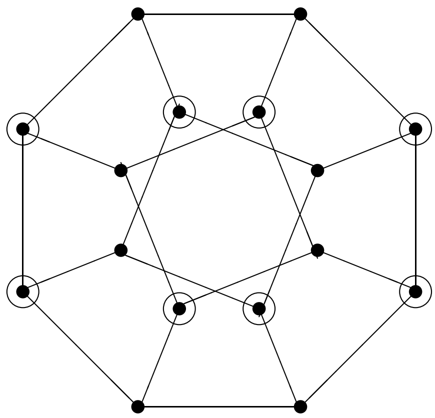

Figure 6: A first attempt shows $t(G(8,2)) \leq \frac{8}{6} = \frac{4}{3}$.

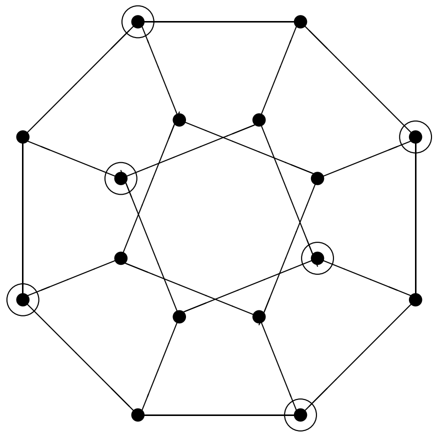

Figure 7: A second attempt shows $t(G(8,2)) \leq \frac{6}{5}$.

In fact, $t(G(8,2)) = \frac{6}{5}$. However, in general, lower bounds are hard to establish.

Trees [10]. If $G = (V, E)$ is a tree with $|V| \geq 3$, then $t(G) = \frac{1}{\Delta(G)}$. Here, $\Delta(G)$ denotes the maximum degree of a vertex of G.

In particular, paths and stars are examples of trees. For $n \geq 3$, $t(P_n) = \frac{1}{2}$. For $n \geq 2$. $t(K_{1,n}) = \frac{1}{n}$.

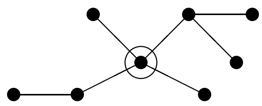

Figure 8: Achieving toughness for trees.

A cycle on n vertices [3]. For $n \geq 4$, $t(C_n) = 1$.

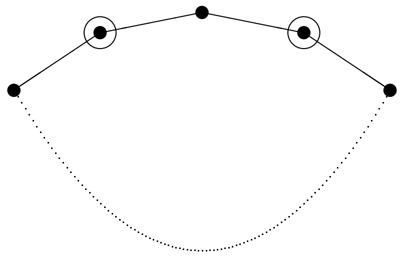

Figure 9: Achieving $t(C_n)$

Compete Bipartite Graphs [3]. If $m \leq n$ and $n \geq 2$, then $t(K_{m,n}) = \frac{m}{n}$.

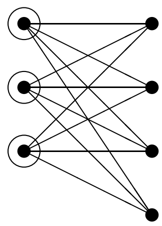

Figure 10: Achieving $t(K_{3,4}) = \frac{3}{4}$

Complete Multipartite Graphs [8]. If $m_1 \leq m_2 \leq \ldots \leq m_r$, then

$$t(K_{m_1,m_2,\ldots,m_r}) = \frac{m_1 + m_2 + \cdots + m_{r-1}}{m_r} \tag{1.3}$$

Multipartite graphs are a natural generalization of bipartite graphs. The vertices of a complete r-partite graph are partitioned into r subsets. Two vertices are connected by an edge if and only if they lie in different subsets. For example, the complete graph on n vertices could be regarded as $K_n = K_{1,1,\dots,1}$. Consequently, some define $t(K_n) = n-1$. However, we are keeping with $t(K_n) = \infty$.

The Petersen Graph $G(5,2)$**.** The Petersen graph has $t(G(5,2)) = \frac{4}{3}$.

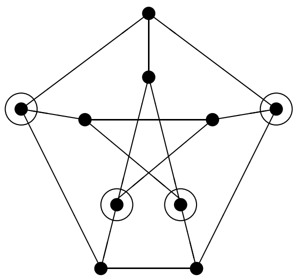

Figure 11: Achieving $t(G(5,2))$

Note that the Petersen graph is an example where a disconnecting set that achieves the toughness is not the same as one that achieves the connectivity.

2 General Results

This section contains general properties of toughness. We start by dispensing with the trivial cases.

Theorem 2.1 ([3]). *Let G be a graph.*

(i) $t(G) = 0$ if and only if G is not connected.

(ii) $t(G) = \infty$ if and only if G is complete.

From now on, only connected noncomplete graphs are considered.

Theorem 2.2 ([3]). $t(G) \leq \frac{\kappa(G)}{2}$

Theorem 2.3 ([8]). $t(G) \geq \frac{\kappa(G)}{\Delta(G)}$

Theorem 2.4 ([3]). *If G is Hamiltonian, then $t(G) \geq 1$.*

Conjecture 2.5 ([3]). *There exists a real number t_0 such that if $t(G) \geq t_0$ then G is Hamiltonian.*

Currently it is only known that $t_0 \geq 2$ if it exists [5].

3 Other Measures of Vulnerability

Some examples of other measures of vulnerabiltiy are listed. Note that, for a given disconnecting set S of vertices, $\tau(G-S)$ is size of the largest component of $G-S$, and $N(S)$ is the set of neighbors of S.

3.1 Integrity [2].

$$min\{|S| + \tau(G-S) : S \text{ is a disconnecting set }\} \tag{3.1}$$

3.2 Tenacity [4].

$$min\{\frac{|S| + \tau(G-S)}{\omega(G-S)} : S \text{ is a disconnecting set }\} \tag{3.2}$$

3.3 Binding Number [12].

$$min\{\frac{|N(S)|}{|S|} : S \text{ is a disconnecting set }\} \tag{3.3}$$

4 My Work

I have studied the toughness of Generalized Petersen graphs which were defined by Watkins [11]. For each $n \geq 3$ and $0 < k < n$, the generalized Petersen graph $G(n,k)$ has vertex set

$$V = \{u_1, \dots, u_n, v_1, \dots, v_n\}$$

and edge set

$$E = \{(u_i, u_{i+1}) | 1 \leq i \leq n\} \cup \{(u_i, v_i) | 1 \leq i \leq n\} \cup \{(v_i, v_{i+k}) | 1 \leq i \leq n\}.$$

Here, all subscripts are taken modulo n. Watkins excludes the case in which $n = 2k$. However, we adopt the convention of Alspach [1] and include those cases as well.

For example, the graph $G(8,3)$ is pictured in Figure 12.

The graphs $G(8,2)$ and the Petersen graph $G(5,2)$ were encountered earlier.

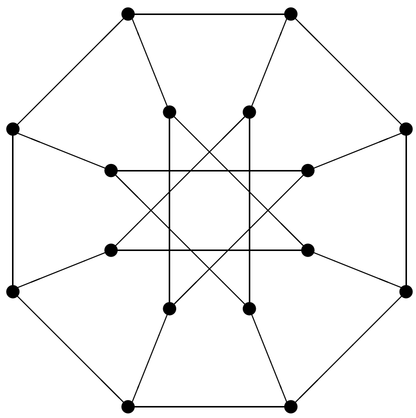

Figure 12: The generalized Petersen graph $G(8,3)$

4.1 Known Results

The toughness of generalized Petersen graphs was first explored in [9]. There, the case in which $k = 1$ is completely settled.

Theorem 4.1 ([9]). *For $n \geq 3$,*

$$t(G(n,1)) = \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n}{n-1} & \text{if } n \text{ is odd} \end{cases}$$

The case in which $k = 2$ is considered in [7]. The value $\frac{5}{4}$ is shown to be the critical value for $t(G(n,k))$.

Theorem 4.2 ([7]). *For $n \geq 5$,*

$$t(G(n,2)) \leq \frac{5}{4} + \epsilon_2(n),$$

where

(i) $\epsilon_2(n) \geq 0$ *for* $n \neq 8$*,*

(ii) $\epsilon_2(n) \longrightarrow 0$ *as* $n \longrightarrow \infty$*, and*

(iii) $\epsilon_2(n) = 0$ *if* $n \equiv 0$ *mod* 7*.*

Theorem 4.3 ([7]). *For $n \geq 5$ and $n \neq 8$,*

$$t(G(n,2)) \geq \frac{5}{4}.$$

Also, $t(G(3,2)) = \frac{3}{2}$, $t(G(4,2)) = 1$, and $t(G(8,2)) = \frac{6}{5}$.

Corollary 4.4 ([7]). *For $n \equiv 0 \bmod 7$, $t(G(n,2)) = \frac{5}{4}$.*

4.2 New Results

The following new results are to appear in [6].

Theorem 4.5. *$t(G(n,k)) \geq 1$ for all n, k.*

Remark 4.6. *Lower Bounds.*

(i) There are infinitely many (n,k) for which $t(G(n,k)) = 1$.

(ii) The lower bound of 1 is thus the best possible.

Our upper bound results are generalizations of Theorems 4.1 and 4.2.

Theorem 4.7. *For k odd and $n \geq 2k+1$,*

$$t(G(n,k)) \leq \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n}{n-(\frac{k+1}{2})} & \text{if } n \text{ is odd} \end{cases}$$

Corollary 4.8. *For k odd and n even with $n \geq 2k+1$, $t(G(n,k)) = 1$.*

Conjecture 4.9. *If n and k are odd and $n \geq 3k$, then*

$$t(G(n,k)) \geq \frac{n}{n - \frac{k+1}{2}}.$$

Theorem 4.10. *For k even, $k \geq 4$, and $n \geq 2k+1$,*

$$t(G(n,k)) \leq \frac{2k}{2k-1} + \epsilon_k(n),$$

where

(i) $\epsilon_k(n) \longrightarrow 0$ as $n \longrightarrow \infty$, and

(ii) $\epsilon_k(n) = 0$ if $n \equiv 0 \bmod 2k$.

Conjecture 4.11. *If k is even, $k \geq 4$, and $n \geq 2k+1$, then*

$$t(G(n,k)) \geq \frac{2k}{2k-1}.$$

4.3 Computer Evidence

To give evidence that our bounds are the best possible, we use a computer to determine $t(G(n,k))$ for all $3 \leq k \leq 7$ and $2k+1 \leq n \leq 18$. Table 1 lists these values of $t(G(n,k))$. The value X signifies that $G(n,k)$ is not defined for those values of n and k. Since $G(n,k) \cong G(n, n-k)$, the table is completely determined by the portion for which $n \geq 2k$.

$k \ddots n$	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3	$\frac{4}{3}$	1	$\frac{5}{4}$	1	$\frac{9}{7}$	1	$\frac{11}{9}$	1	$\frac{13}{11}$	1	$\frac{15}{13}$	1	$\frac{17}{15}$	1
4	$\frac{5}{4}$	$\frac{5}{4}$	$\frac{5}{4}$	1	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{11}{9}$	$\frac{5}{4}$	$\frac{13}{11}$	$\frac{14}{11}$	$\frac{5}{4}$	$\frac{8}{7}$	$\frac{16}{13}$	$\frac{6}{5}$
5	X	1	$\frac{5}{4}$	1	$\frac{4}{3}$	1	$\frac{9}{7}$	1	$\frac{13}{10}$	1	$\frac{5}{4}$	1	$\frac{17}{14}$	1
6	X	X	$\frac{7}{6}$	$\frac{6}{5}$	$\frac{9}{7}$	$\frac{5}{4}$	$\frac{9}{7}$	1	$\frac{9}{7}$	$\frac{7}{6}$	$\frac{5}{4}$	$\frac{16}{13}$	$\frac{17}{15}$	$\frac{5}{4}$
7	X	X	X	1	$\frac{4}{3}$	1	$\frac{11}{9}$	1	$\frac{9}{7}$	1	$\frac{13}{10}$	1	$\frac{17}{14}$	1

Table 1: Toughness Values

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