Optimal Networks Have Maximum Toughness

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Abstract

Given fixed numbers of computers and cables to connect them, what is the best way to link up the computers to form a network? We consider graph parameters which measure the strength of a network.

1 A Motivating Example

Given 6 computers and 5 cables to connect them, what is the best configuration in which to connect them to form a network? Figure 1 displays all of the possible connected configurations. Note that we represent computers by points (vertices) and cable connections by lines (edges) joining pairs of points. That is, networks can be represented by graphs. Consequently, Figure 1 displays all of the possible connected graphs on 6 vertices and 5 edges.

Of course, to determine the best configuration in Figure 1 we need to know what we mean by *best*. The criteria that we use here is that we want to minimize the potential of destruction of the network by

1. computer failures

2. computer sabotage

In fact, since we really want to avoid worst case scenarios, we should primarily keep the possibility of sabotage in mind. That is, we should worry that the computers that we would least want to fail will be the ones that do fail.

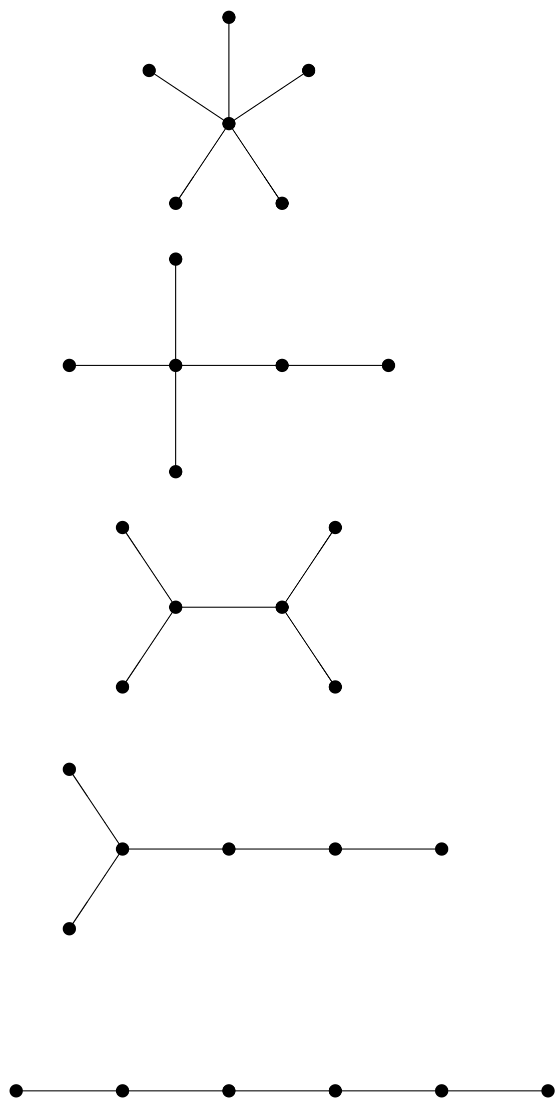

Figure 1: Some $(6, 5)$-graphs

So how would the saboteur decide which computers to make fail? That is, where would a saboteur find weakness in a network? From the saboteur's point of view, the weakness of a network is determined by

1. The minimum number of faulty computers which will cause the network to be fractured into multiple pieces (disjoint subnetworks).

2. A "bang for his buck" measure which considers the ratio of the number of faulty computers to the number of resulting disjoint subnetworks. The saboteur wants this ratio to be as small as possible.

Condition (i) is reflected by the *connectivity* [4] of the corresponding graph. Condition (ii) is reflected by the *toughness* [1] of the corresponding graph.

2 Definitions

Let G be a graph with vertex set V. For any subgraph H of G, let $\omega(H)$ denote the number of components in H.

The *connectivity* of G is

$$\kappa(G) = \min\{|S| \ : S \subseteq V, \omega(G–S) > 1\}.$$

The *toughness* of G is

$$\tau(G) = \min\{\frac{|S|}{\omega(G–S)} : S \subseteq V, \omega(G–S)>1\}.$$

Very simple, the connectivity of a graph is the smallest number of vertices that one would need to remove to leave the graph disconnected. We define connectivity very formally above to emphasize the similarity of that notion with the notion of toughness, which is less suited to an informal definition. In Figure 2, the connectivity and toughness of the graphs from Figure 1 have been computed.

The central issues are now coming into focus. The saboteur hopes for low connectivity and low toughness. The network designer thus aims for maximum possible connectivity and/or maximum possible toughness. If we fix the number of vertices (computers) and the number of edges (cables connecting them), then what is

1. the most highly connected graph?

2. the toughest possible graph?

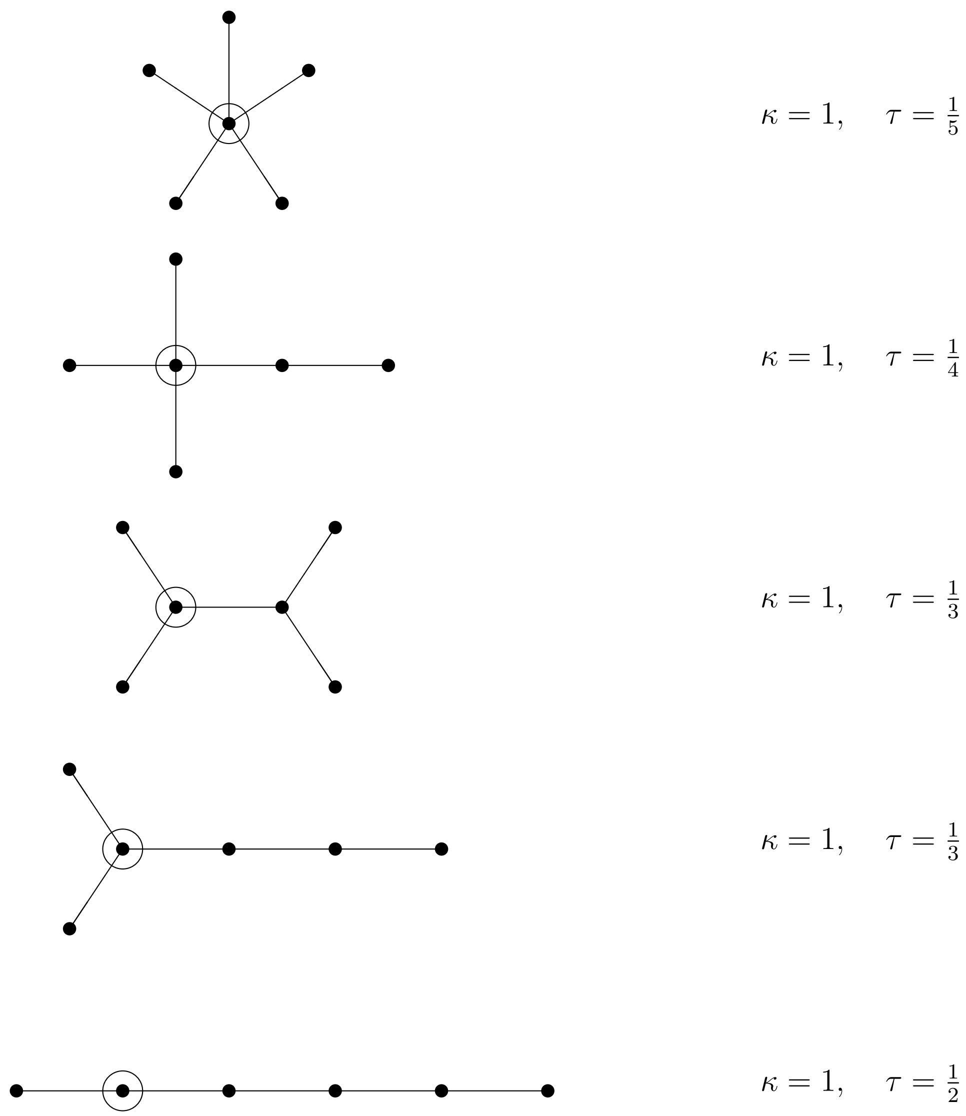

Figure 2: Vulnerability of $(6, 5)$-graphs

Of course, we have converted this into a graph theory question. We are fixing the number of vertices, say $|V| = n$, and the number of edges, say $|V| = n$. We call any graph with n vertices and m edges an (n, m)-graph.

We want (n, m)-graphs with maximum possible connectivity and maximum possible toughness.

Maximum connectivity is

$$C_n(m) = \max\{\kappa(G)\colon G \text{ is an } (n{,}m)\text{-graph}\}.$$

Maximum toughness is

$$T_n(m) = \max\{\tau(G)\colon G \text{ is an } (n{,}m)\text{-graph}\}.$$

An (n, m)-graph G is *maximally connected* if $\kappa(G) = C_n(m)$ and *maximally tough* if $\tau(G) = T_n(m)$.

3 A Typical Example

Consider $n = 6$ and $m = 10$. That is, consider graphs on 6 vertices and 10 edges. What are $C_6(10)$ and $T_6(10)$? Figure 3 displays not all $(6, 10)$ graphs, but probably those which one would consider the most reasonable choices for achieving maximum connectivity and maximum toughness. Which of the graphs there seems to be the best?

In Figure 4 the connectivity and toughness of the graphs from Figure 3 have been computed. In fact, a single graph that is both maximally connected and maximally tough can be seen there. The maximum possible connectivity is 3. That is $C_6(10) = 3$. The maximum possible toughness is $\frac{3}{2}$. That is $T_6(10) = \frac{3}{2}$.

4 General Examples

Notice that $T_6(5) = \frac{C_6(5)}{2}$ and $T_6(10) = \frac{C_6(10)}{2}$. In fact, this is suggested of something general. In most cases,

$$T_n(m) = \frac{C_n(m)}{2}$$

In general, however,

$$T_n(m) \leq \frac{C_n(m)}{2}$$

This relationship between $T_n(m)$ and $C_n(m)$ useful to us since Harary [3] computed $C_n(m)$.

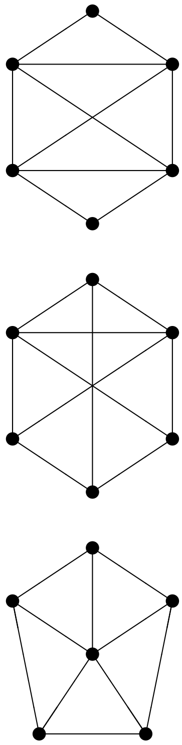

Figure 3: Some $(6, 10)$-graphs

Theorem 4.1 (Harary '62). *For* $n \geq 1$ *and* $0 \leq m \leq \frac{n(n-1)}{2}$,

$$C_n(m) = \begin{cases} 0 & \text{if } m \leq n-2, \\ \lfloor \frac{2m}{n} \rfloor & \text{if } m \geq n-1. \end{cases}$$

Harary proved Theorem 4.1 by explicitly constructing graphs which achieve the asserted values of maximum connectivity. Those graphs have come to be known as *Harary Graphs.* We denote the Harary graph on n vertices and m edges by $H(n, m)$. It is also convenient to introduce the additional variable $r = \lfloor \frac{2m}{n} \rfloor$.

Consider first when $m = \lceil \frac{nr}{2} \rceil$. An example of Harary's construction when r is even is displayed in Figure 5. When r is odd, the construction depends on the parity of n. An example when n is even is displayed in Figure 6 and one when n is odd is displayed in Figure 7. For a general number of edges m,

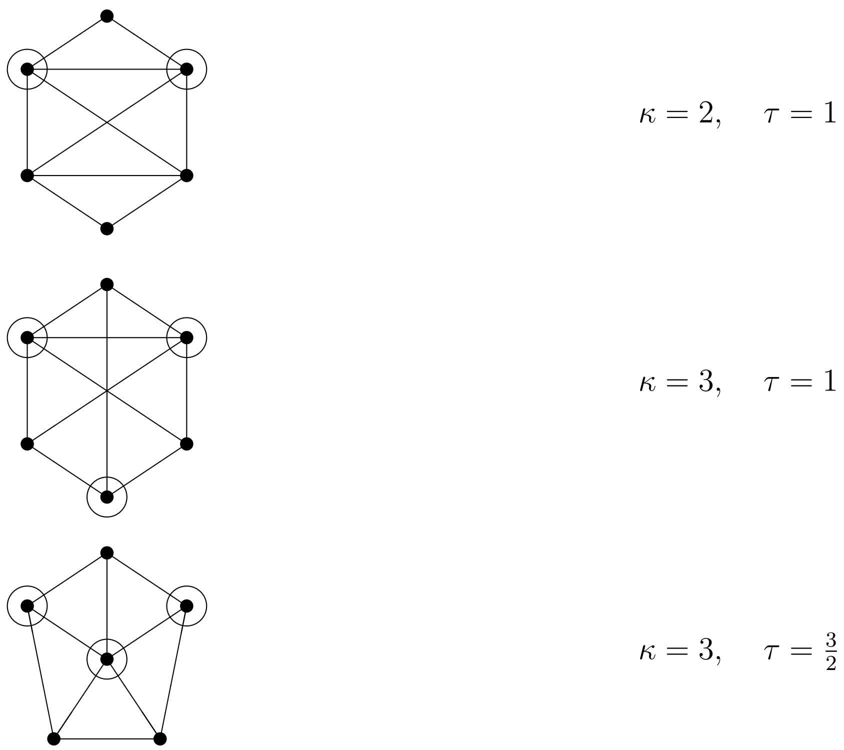

Figure 4: Vulnerability of $(6, 10)$-graphs

we simply add $m - \lceil \frac{nr}{2} \rceil$ edges to the Harary graph with $\lceil \frac{nr}{2} \rceil$ edges. Theorem 4.1 follows from the fact that Harary graphs are maximally connected.

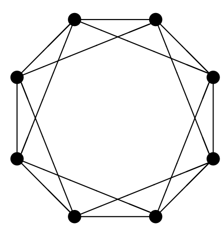

Figure 5: r even; e.g. $H(8, 16)$ so $r = 4$

In general, Harary graphs are not maximally tough. However, many of them are. In particular, toughness behaves nicely when r is even.

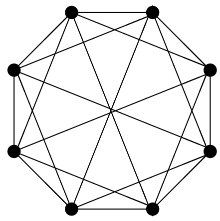

Figure 6: r odd and n even; e.g. $H(8,20)$ so $r=5$

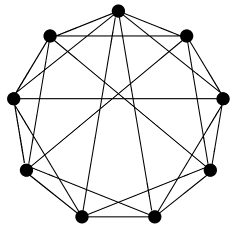

Figure 7: r odd and n odd; e.g. $H(9,23)$ so $r=5$

Theorem 4.2 (Ferland '02). *If r is even and $\lceil \frac{nr}{2} \rceil \leq m < \lceil \frac{n(r+1)}{2} \rceil$, then $T_n(m) = \tau(H(n,m)) = \frac{r}{2}$.*

Toughness is less well-behaved when r is odd. We need some extra constraints on m to guarantee that the appropriate Harary graphs are maximally tough.

Theorem 4.3 (Ferland '02). *If $m \geq n \left(2\lfloor \frac{n}{6} \rfloor + \lfloor \frac{n \bmod 6}{3} \rfloor\right)$, then $T_n(m) = \frac{C_n(m)}{2}$.*

Note that the hypothesis of Theorem 4.3 is equivalent to

$$r = \lfloor \frac{2m}{n} \rfloor \geq 2 \left(2 \left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n \bmod 6}{3} \right\rfloor \right).$$

That is, the assumption is that roughly $\frac{2}{3}$ of all possible edges are present.

Extending the ideas of Harary we define *Sub-Harary Graphs* $H'(n,m)$. These are constructed if $r = \lfloor \frac{2m}{n} \rfloor$ is odd. Very simply sub-Harary graphs are obtained from the Harary graph $H(n, \frac{n(r+1)}{2})$ by removing up to $\frac{r+1}{2}$ well-chosen edges. Figure 8 displays the sub-Harary graph $H'(8,14)$, which we obtain by removing a certain 2 edges from the Harary graph $H(8,16)$.

Sub-Harary graphs also turn out to be maximally tough.

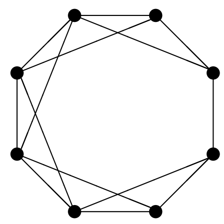

Figure 8: $H'(8, 14)$ so $r = 3$

Theorem 4.4 (Ferland '02). *If* r *is odd and*

$$\frac{(n-1)(r+1)}{2} \leq m < \frac{n(r+1)}{2},$$

then $T_n(m) = \tau(H'(n, m)) = \frac{r}{2}$.

5 Difficulties

For $n = 7$ and $m = 11$, The Harary Graph $H(7, 11)$, displayed in Figure 9, is maximally connected (with connectivity 3) and has toughness 1.

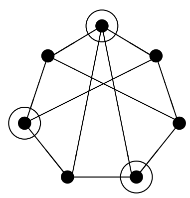

Figure 9: $\tau(H(7, 11)) = 1$

However, the $(7, 11)$-graph displayed in Figure 10 has toughness $\frac{4}{3}$. So $H(7, 11)$ is maximally connected but not maximally tough. In fact, the graph in Figure 10 is maximally tough. So, $T_7(11) = \frac{4}{3}$. We therefore have a case in which $T_7(11) < \frac{C_7(11)}{2}$.

The central difficulty is that

G maximally connected $\not\to$ G maximally tough.

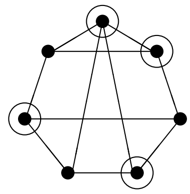

Figure 10: $\tau = \frac{4}{3}$

An open question is

G maximally tough $\rightarrow$ G maximally connected?

References

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