Teaching Mathematics Using the Genetic Approach

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Introduction

Instructors have used the history of mathematics in the classroom in many different ways. Look at a collection of calculus textbooks. Usually, history is confined to picture captions or marginal notes. It serves merely as amusement or ornament. Or you may find a few page at the end of the chapter. This allows some indication of the cultural background to some discovery. Occasionally, the history of mathematics is used as a source of examples or exercises. Is this all that the history of mathematics has to offer a mathematics instructor?

I suggest instead that we let history answer the question, How and why did this method (formula, theorem) come about? Talking through a sequence of such questions and answers is called teaching by the genetic approach. In his book, *The Calculus: a genetic approach*, Otto Toeplitz gave the best and largest-scale example. To illustrate, I shall now discuss a strand of ideas from the calculus. Toeplitz discussed these too, but in a slightly different way.

Example

Students used to learn a whole bagful of tricks under the heading of "techniques of integration." It takes some stepping back, which is *not* usually encouraged by textbooks, to realize that there are only a few basic ideas. Once one knows that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$$

$$\int x^{-1} dx = \ln|x| + c$$

$$\int e^x dx = e^x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \frac{1}{1+x^2} dx = \arctan x + c$$

one can obtain all the integrals (in the book) by

1. sums and scalar multiples
2. products (*i.e.* integration by parts)
3. substitution.

Rational functions form a fundamental class of example that use method 1. Every proper rational function can be decomposed[1] into partial fractions, and since there are only a few kinds of partial fractions, we can exhaust the possibilities of integration. We must use integration by parts to give us appropriate feductions formulas like

[1] Because the partial fraction decomposition depends upon the factoring of the polynomial denominator, we have a technique for the integration of rational functions sometimes only *in principle*.

analytic geometry gives the expressions above for the coordinates of that point. The same picture also tells us how trigonometric substitutions arise. They apply to integrals of the form

$$\int R(x,p(x))dx,$$

where R is, as before, a rational function of two variables, and $p(x)$ is a polynomial of degree two.

Is this worthwhile? In a practical, computational sense, No. Give yor hard integrals to Maple or Mathematica. But in terms of understanding and motivation, it is wonderful! It is wonderful firstly, because it shows coherence where students usually see a bag of tricks. More important, it provides an antidote to the unspoken message of those first lessons on integration, that all functions can be integrated (in elementary terms) if only we are clever enough. The upper semicircle is, of course, a curve with equation

$$y = \sqrt{p(x)},$$

where $p(x)$ is a polynomial of degree two. If, instead of a polynomial of degree 1 or 2, we introduce a polynomial (without square factor) of degree 3, the resulting integral is no longer elementary. We have an elliptic integral. (It is worthwhile to discuss the simple pendulum to underline the value of these integrals.) Of course, a great deal of nineteenth century analysis began with these integrals.

Why, for these elliptic integrals, is there no rational substitution like that of Euler? Algebraic geometry answers our question: the curves now are no longer rational, or unicursal. My point is that a problem in the calculus leads to a question in geometry. (Incidentally, a good exercise in analytic geometry is to derive the substitution, analogous to the Euler substitution, for the equilateral hyperbola, and thereby to show (again) that all rational functions of the hyperbolic functions are integrable (in elementary terms).)

By the way, in a certain sense the Euler substituion goes back at least to Diophantus, as a formula for producing all Pythagorean triples. Again, we may note that the sperical analogue of the Euler substitution yields the stereographic projection of the sphere, which is of inportance in cartography as well as geometry and complex analysis.

Conclusion

Treating the techniques of integration in this way conforms to that poetic ideal of unity in variety. The *variety* here includes possible discussions of existence theorems, algorithms, theory of equations, functions of a complex variable, number theory, and algebraic and differential geometry. By providing opportunities for fruitful digression, the approach enriches the calculus course, allowing prospective mathematics majors intriguing glimpses of their subject. In this way, the calculus, which is still the most frequent introductory mathematics major course, can enjoy one of the advantanges of the introductory physics course, which usually surveys the subject.

On the other hand, the presentation that I have outlined has *unity*. Thge entire presentation responds to the question, What functions can be integrated in elementary terms?